teacher's guide

math workshop for children

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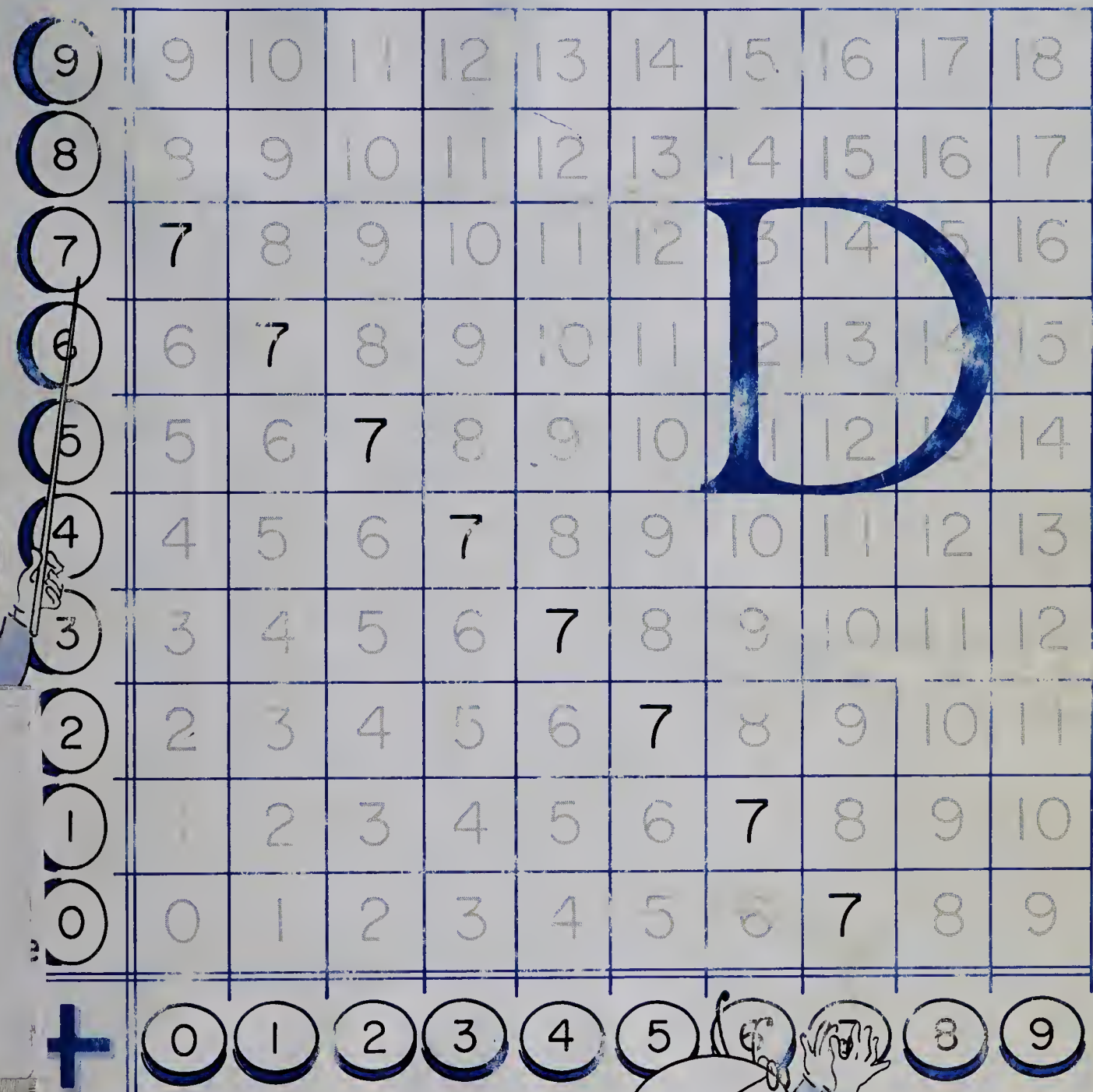

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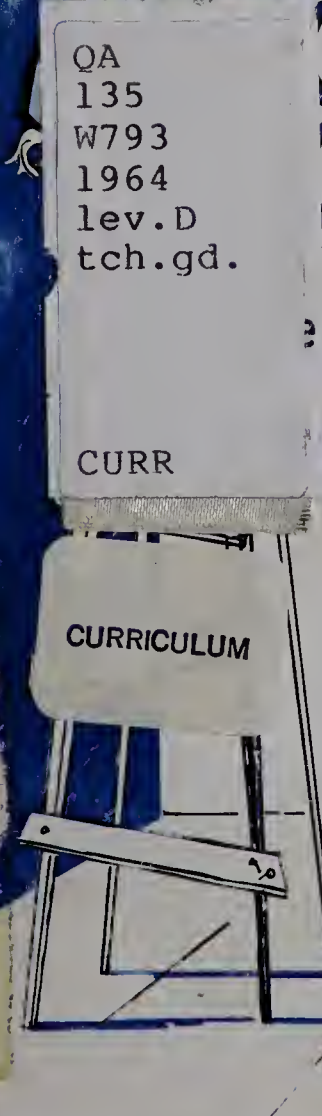

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LEVEL D
TEACHER'S GUIDE
MATH WORKSHOP FOR CHILDREN

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Preface

Work Text Level D, as all levels of *Math Workshop* materials, has been prepared with the knowledge that many children will not have been introduced to anything other than traditional approaches to arithmetic. No familiarity with *Math Workshop* materials is expected of them. At the same time, children who are familiar with the *Math Workshop* will not find repetitions of previous work. Rather, activities have been designed that are basically new to all children.

Further, *Level D* starts with few assumptions of prior work in arithmetic, but moves steadily through all the content generally expected of children in about Grade 4 on standardized achievement tests. Thus, children can transfer into and out of the work without uncomfortable feelings of being placed at a disadvantage.

Spiral curriculum

The concept of the spiral curriculum, as outlined by the *24th Yearbook of the National Council of Teachers of Mathematics,* has been used throughout all levels of the *Math Workshop.* At each level, from the first on, the main ideas of mathematics have been introduced. Each successive year's activities deal with these unifying ideas with increasing sophistication. While "multiplication," for example, was introduced in *Level A,* it was reintroduced in *Level B* and quickly raised to a more advanced kind of activity; similarly, with *Level C.*

In *Level D,* the idea is introduced once again, beginning with pages 6 and 7, with the "row-column" concept ("array" or "Cartesian" product as some call it), while a thinly veiled "repetitive addition" approach is taken on pages 10 and 11. Through the device of a special form of the Cross Number Puzzle, the traditional algorism is anticipated.

"Ben's Story" begins on page 20 of *Level D.* With very few exceptions, children do not recognize that Ben's "strange" method rests squarely on exactly the same mathematical ideas as the traditional algorism. As the story develops, there is drill on the basic facts, but the student's attention is focused on trying to find out why Ben's method always works. And by pages 30-31, the children are handling two-place factors with a deep understanding of what they are doing.

This is an example of the way in which each fundamental idea is, at each level, reintroduced in a fresh way, reinforced, and rapidly carried to a considerably higher level of understanding. Addition, subtraction and multiplication of common fractions, decimal fractions and directed numbers are encountered—as if for the first time—and developed to varying levels of understanding.

Less than mastery

However, children are not expected to "master" such elusive ideas as multiplication of fractions or addition of positive and negative numbers—but they will meet all such ideas again and again, building always more layers of understanding.

The spiral curriculum rejects the "building-block" approach to arithmetic which has been used in the past to structure scope and sequence charts. Rather, the spiral curriculum calls for the early introduction of fundamental and unifying ideas, for repeated reconsideration at later times, continually nurturing the important concepts year after year as understanding develops.

The hope for "mastery" must be limited to that part of arithmetic that must be committed to memory—the vocabulary of arithmetic that must be instantly retrievable—the basic facts of arithmetic. Here, mastery can and must be achieved. But where big ideas are involved, mastery does not come in this way. It develops slowly and is based completely on understanding.

Approaches to individual activities

There are at least three generally successful ways in which individual pages or groups of pages can be handled in the classroom:

1) Use the *Work Text* to conclude a development;
2) use the page as the basis for discussion;
3) let children work independently.

1) In those areas in which the teacher has many ideas of his own, there is no substitute for direct teaching . . . with liberal use of manipulative material, the blackboard and class discussion. The *Work Text* is then used late in the development to tie down, or simply to record the progress that has been made to deeper understanding.

Page 13 of *Work Text Level D* can be used as an example. The basic idea here is that *numbers have many names.* There are limitless ways to initiate lively activities around this important and useful concept. For instance, how many names are there

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for 12 in the form of ___ + ___ if only odd numbers can be used in the blanks? How many if only even numbers are considered? How many if we consider only multiples of 3? How many if we consider only numbers with a remainder of 1 when divided by 3? (Impossible—only 1, 4, 7, and 10 can qualify and the sum of no pair is 12). Change the form to ___ + ___ + ___. How do the answers to the same questions change?

A game that anticipates the sequences of pages to follow is to ask similar questions, but add the rule that "one of the blanks must be filled in with a multiple of 10." This, of course, also anticipates one very important use of regrouping in all the main algorisms!

After such preliminary work, class members are ready to move through page 13 in the final few minutes of the arithmetic, needing no help at all.

2) Pages 20 and 21 might, in contrast, be gone through very carefully with an entire class. There is a light story line and many opportunities for humor. Ben seems to know so little—no multiplication beyond 5×5—and yet he can find the product of 8 and 6.

"Is there any limit to Ben's little trick which starts out with 'regrouping'? you might ask. But could he handle $5 \times 127 =$ ___? How would you handle that same problem? Do you prefer Ben's method? Why is it clumsy?

"Do you think Ben can handle division?" (Of course, we can look ahead on page 21 to see that he at least tries.) "Can you anticipate how Ben will find the result in $24 \div 4 =$ ___ without knowing that $4 \times 6 = 24$?"

Such a page or sequence of pages offers many opportunities to dig deeply into "what's going on here?" And, because we are focusing attention on understanding the algorism of multiplication, such pages deserve considerable time and effort.

3) A third basic method of approach might be taken with page 38, for example. "Open your *Work Text* to page 38. Look over the page. Think about it. See if you can decide for yourself what it is that you are being asked to do.

"See if you can find out all by yourself. Wait a while before you ask for help. And, let's make today a 'quiet day.'"

Now, the teacher is supposedly reluctant to help but, of course, is watching carefully for those who haven't sufficient self-confidence to make an effort. When several children are unable to get started, suggest to two of them that they help each other quietly. Usually, this causes a change in attitude that leads to a new effort. Few, if any, will require much help in getting underway.

The activity on page 38 is scaled in two directions—it gets more difficult as you move to the right on the page, and down the page. The teacher may let time run out, suggesting that those who haven't finished might like to finish the page later . . . or take it home to finish. Or, perhaps those who were able to bridge the gaps and see the patterns in some of the last five series should be asked to give others help by telling what the next few entries would be in each line . . . but urged not to give away "the secret." Thus, independent work is encouraged.

Individual differences

There are two sources of material designed for the more able children. First, in this guide, many suggestions are given for simple but mathematically rich activities that can be outlined quickly to individuals or groups of individuals for independent work. Second, the last section of the *Work Text,* beginning with page 100, may be reserved for only the more able members of the group. It is built around a very informal and light approach to some of the early ideas in algebra and geometry. That portion of this guide that accompanies pages 100 to 128 might very well be given to some of the children who want to work through that section of their *Work Text* by themselves. Perhaps they can play at the role of teacher for others who run into snags along the way at this point.

Supplementary *Work Forms*

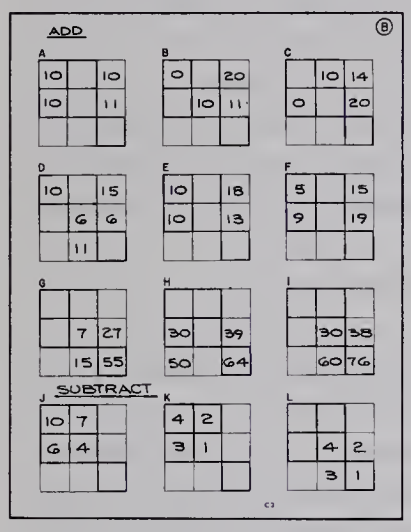
Many of the interesting and versatile page-forms developed in *Math Workshop* texts have been extracted and made available as Ditto and Multilith masters called *Work Forms.* In a few minutes, a teacher can convert one of these forms with pen or pencil into a rich supplementary activity for the whole class or for a groups within the class. Providing, as they do, significant, independent learning activities for creative seatwork and meaningful homework, and taking into consideration, as they do, the wide range of individual abilities within any class, the *Work Forms* should prove very useful to busy teachers.

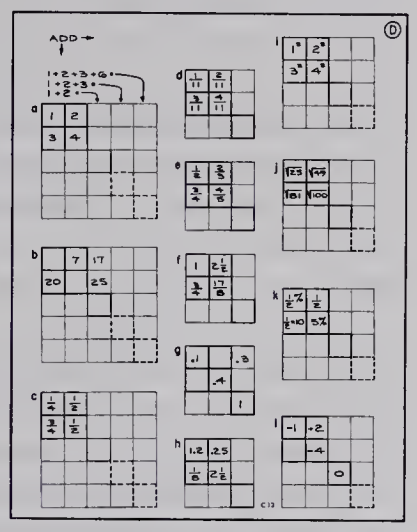
The 59 masters include:

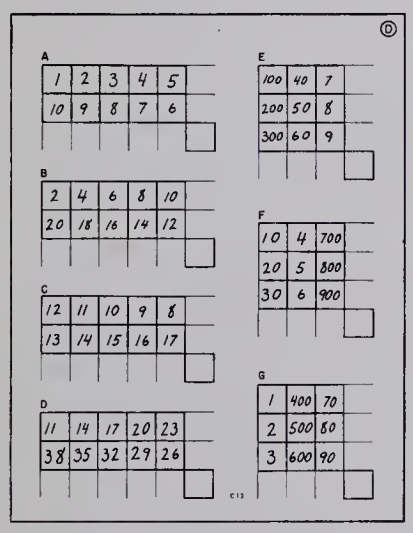
10 Number Line Forms, numbered N1 to N10
14 Cross Number Forms, numbered C1 to C14
35 General Forms, numbered G1 to G35

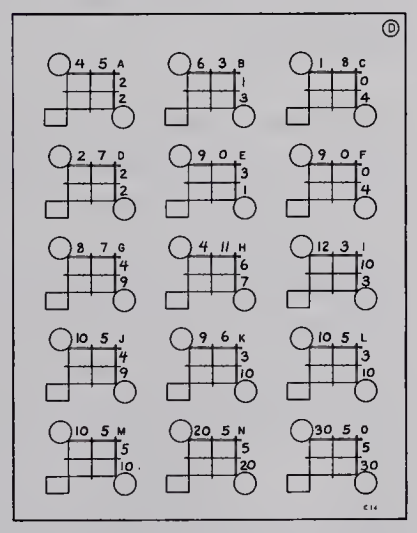
Many of the examples cited throughout this teacher's guide refer to the *Work Forms* (which are themselves accompanied by a guide). For example, the first nine pages of *Work Text Level D* are an introduction to Cross Number Puzzles. While the children are being drilled in the basic facts of arithmetic, they are really concerned with the puzzle: Why does it work? And under what conditions will it work? Among the *Work Forms* are several versions of Cross Number Puzzle blanks that can be used, without any conversion by the teacher, as pages on which the children can make up their own examples.

In reduced size, here are a few examples of the Cross Number Puzzle forms that are available in 8 1/2 × 11-inch actual size:



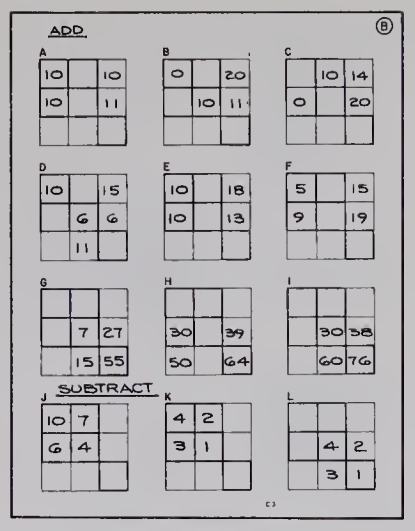


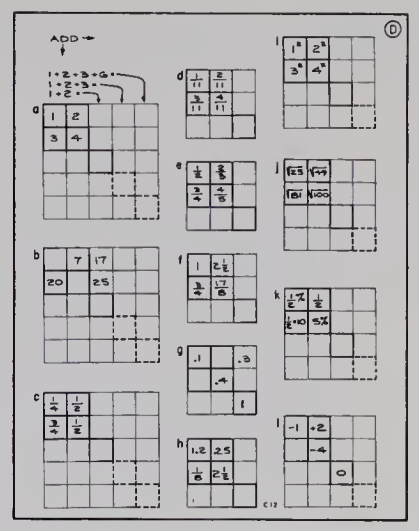


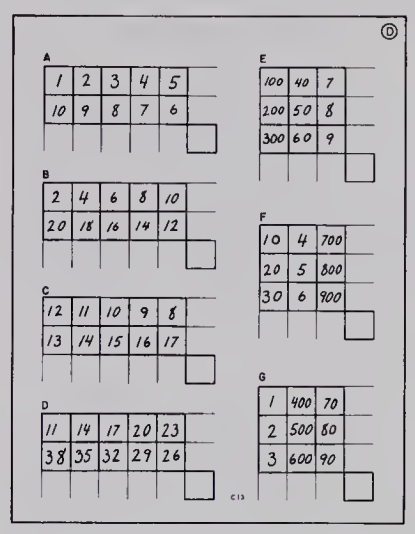


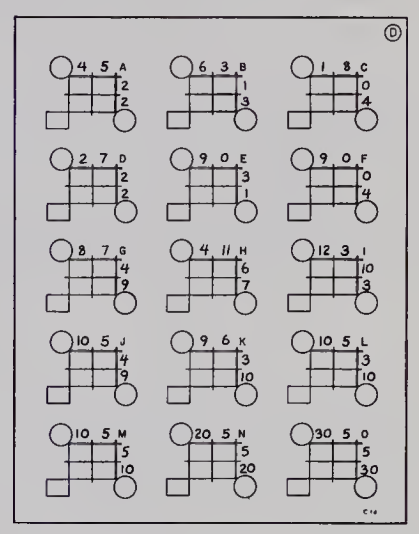
The *Work Text* itself suggests many ways in which these forms can be used to help develop important ideas (and, at the same time, provide essential drill in the fundamental facts). Teachers can create similar activities by adding whatever numbers they want in a matter of minutes.

The *Work-Forms Guide* has, of course, many suggestions for ways in which these *Work Forms* can be put to use. Here are some examples from the *Work-Forms Guide*—with answers shown in red:

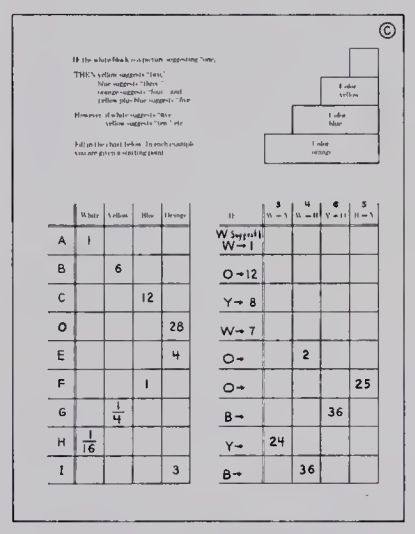








Here are some other examples of *Work Forms* that you will recognize as you think through the children's pages for *Level D:*



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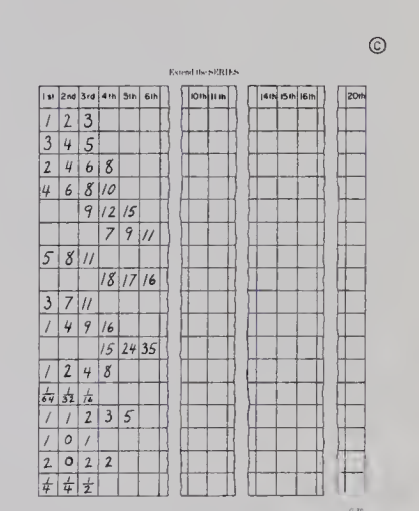

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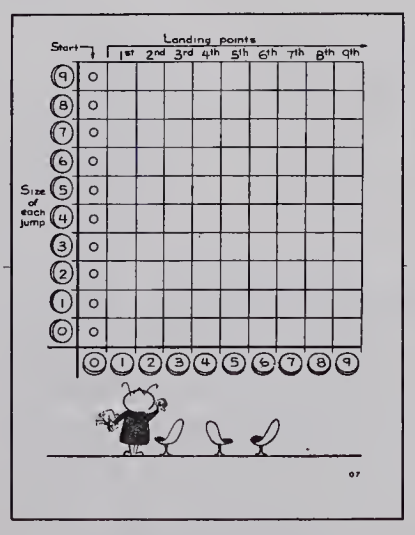

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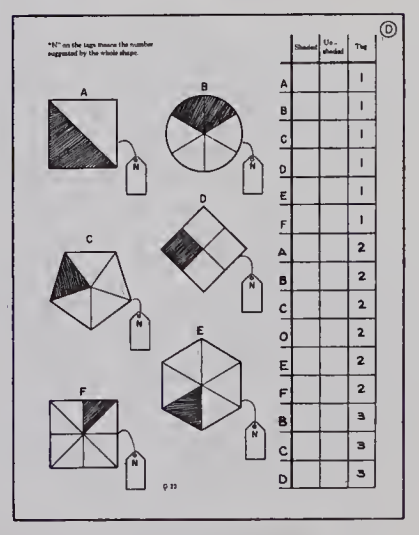

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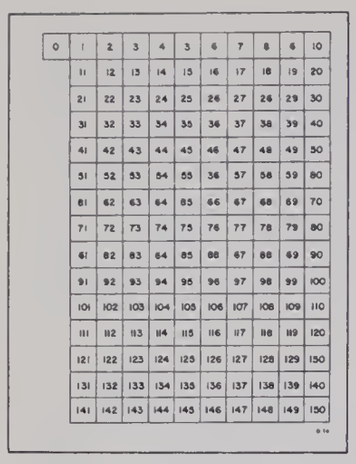

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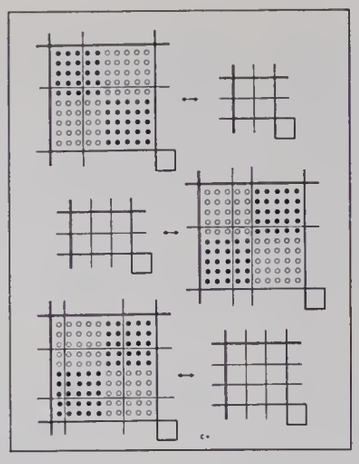

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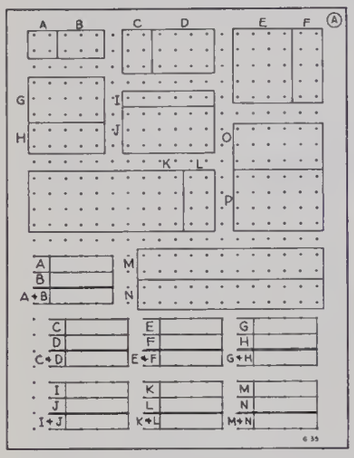

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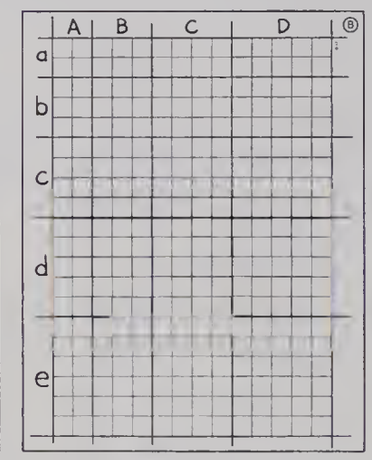

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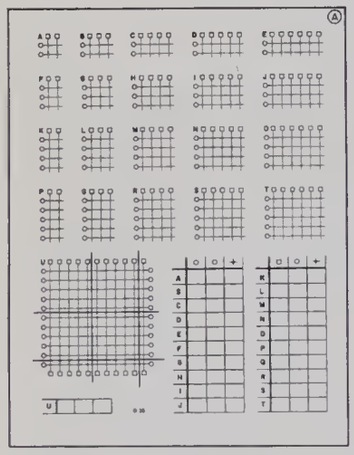

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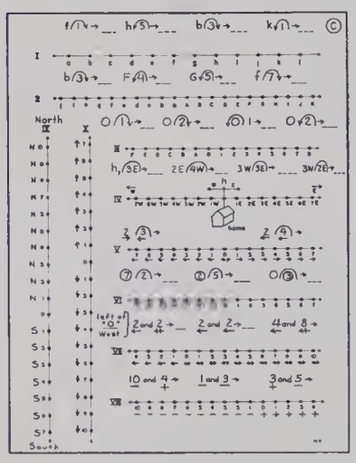

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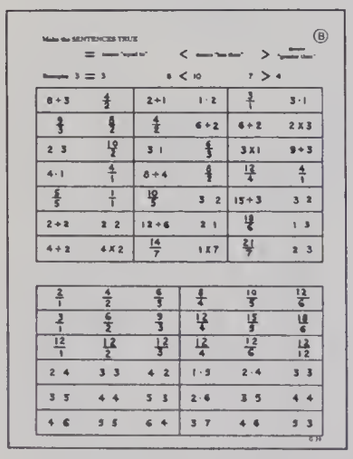

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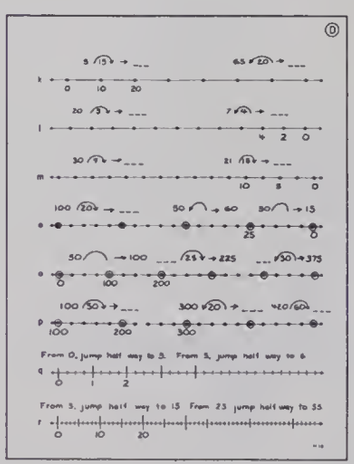


See page 60

Countless other uses for the *Work Forms* can be found in the children's work texts and the teacher's guides to the other levels of the *Math Workshop*. The ideas in the particular adaptations found there are easily adaptable to any level of development.

Mastery Tests

The set of 59 *Work Forms* contains a battery of six tests, each of which presents all the basic facts of addition, subtraction, multiplication and division. An answer sheet is provided so that children can check their own or their neighbor's work. In addition, an individual record sheet enables each child to chart his own progress in terms of accuracy and time required to recall all the basic facts.

The *Work Forms*, then, provide the teacher with a flexibility and versatility never before available in an organized and integrated program. While they are not essential to the use of the *Math Workshop* texts, they are an invaluable supplement, an effective means of meeting individual needs, and a help to every teacher in expressing his own individual approach to helping children understand arithmetic.

Practice in recall of basic facts

While almost every page of *Work Text Level D* requires many responses that depend on recall of basic facts, the focus is elsewhere on a development that has direction and a goal. The 'basic facts' are, in a sense, a means to an end—and are thus recognized by children as the essential vocabulary of arithmetic.

However, beginning on page 26, an additional device is introduced that, at first, asks directly for a series of responses to the equivalent of $3 \times 7 =$ ____, $3 \times 9 =$ ____, $3 \times 6 =$ ____, etc. See also the exercises that stretch across the middle of page 26, the top of page 27, the top and bottom of pages 28, 29, 30 and 31. Notice the variation in the exercises at the bottom of pages 28 and 29. A series of products is given. In the top of the two rows at the left you will note "___x" which, according to the pattern of earlier similar exercises, asks for a factor that is common to all the products. Since 1 could always be a common factor to introduce a bit of the "game element," it is best to establish the rule that "___x" must be filled in by noting the *largest* common factor in all the products given.

In the example at the bottom of page 28, the student is asked to find the common factor of a series of products that includes 24, 32, 36, 28, etc. Clearly, 1 and 2 are common factors of these products; but so is 4. And while 8 is common to the first two products given, it fails to qualify as a factor of 36. Thus, it is soon clear that 4 is the greatest common factor.

A further variation is introduced at the bottom of page 31. In some examples, only one factor is given and, in others, only the product is given .

At the bottom of page 33, the game becomes a trifle puzzling. The first five examples are fully or partially completed to help explain the rules of the game. The sixth example asks the student to find two numbers which, when thought about as addends, lead to a sum of 17; but, also, if thought about as factors, lead to a product of 30. It takes a lot of thinking to come up with the correct response of 10 and 3 (or 3 and 10—order clearly makes no difference in this exercise).

A question that should be asked continually in this game is: "Can you find another pair of numbers that satisfy these conditions?" (Never tell, but the pair is always unique—there is always one and only one pair of satisfactory numbers.)

Unless we continue the rule that 1 is considered as a factor that is admitted as a response only when necessary, the examples on pages 34, 40, 41, etc., would be ambiguous. All the information given about one pair of numbers on page 34 is that, taken as factors, their product is 27. Without the rule, 1 and 27 would qualify equally as well as 3 and 9.

(If the class is familiar with the use of letters to express the idea of patterns, they would see that this exercise can be explained as follows:

a + b			
a			
b			
a · b			

(The game can be made more challenging in this case by expanding the pattern in one of countless ways, such as:

2m + n			
m + n			
m			
n			
m n			
m(n + 1)			

(If some care is taken in the selection of expressions, filling in any two cells in a row provides sufficient clues to find the numbers that must be indicated in all other cells.)

Problem solving

Many people use the expressions "problem solving," "word problems," and "story problems" interchangeably in referring to such written questions as "Johnny lost 7 of his 11 pennies. How many has he left?"

Another example: "Farmer Brown had 8 posts. Each weighed 10 pounds. How much did the posts weigh altogether?" The answer, of course, is 80 pounds because $8 \times 10 = 80$ . . . or is it $10 \times 8 = 80$?

A slight change in the statement transforms it into a bit of a problem—into more than an example. "Farmer Brown had 8 posts. He set them out in a straight line, and 10 feet apart. How far is it between the end posts?"

To answer the question, you had better not look for "cue" words. Think for a moment; visualize or make a sketch of the situation. Once you understand the essence of the problem, then only is it reduced to an example from life of the pattern that "$7 \times 10 = 70$."

Now, both "examples" and "problems" have a vital role in helping children understand ways in which arithmetic applies in the physical world, but let us explore the use of "examples" in the teaching of elementary school arithmetic.

Professor W. W. Sawyer tells about a class of able fourth grade students that was asked: "Where would you ever use the fact that $24 \div 4 = 6$?" After considerable thought, a boy replied: "I might use it to answer some questions at school." "Yes," Professor Sawyer admitted, "but where else?" "Well," another student said after deep thought, "My mother might ask me to go to the store and get her $24 \div 4$ eggs."

"My mother never did that to me," was Professor Sawyer's reply.

Such episodes suggest a school program in which "examples" are not used as effective tools to help develop better understanding of arithmetic.

"Examples," when properly used, can be, so to say, the first baseman who clinches a very effective triple-play: from manipulative materials-to blackboard-to examples . . . examples from life.

The play begins in the world of objects children can feel and see and manipulate. A pattern is noted in otherwise unrelated events. Sketches and diagrams on the blackboard can help focus attention on the aspect of the experience that has been singled out for consideration. We need to be able to refer to a pattern we have noticed, so we attach a symbol or group of symbols as a name for the pat-

tern and thereby increase our command of the new language of arithmetic. This is an act of abstraction. We have given a name to a pattern.

Can each child use this new bit of language? Can he tell us about examples in which he recognizes this same pattern?

"Tell us a story. . . ." If the child can use his new language in this way, we've accomplished our immediate goal. If he can't, we must go back to the world of objects at hand and try again.

The child's response at the vital story-telling stage is the most reliable evidence a teacher can have that a satisfactory level of understanding has been reached. Without this immediately available evidence, the teacher could unwittingly leave some children lost in the strange world of abstract symbols—and that, for a child, is a chilling experience.

Children are brought to school by proud parents who tell us their Mary can count all the way to 25. But when she counts the fingers on her two hands, she ends up with the surprising conclusion that she has 13 fingers!

Other children are described as already knowing the addition facts. See for yourself: "Billie, how much is 4 plus 3?" He performs with an appropriate "7." But later we talk to Billie about a party where four children were sitting at a table, and then three more children arrived. "How many were there then?" He doesn't know.

So every good teacher knows he must find out what the language of arithmetic a child uses actually means to the child.

Most children, by this time, have learned that $4 \times 6 = 24$. They know that in 4 rows of checkers with 6 in a row, there are 24 checkers. If necessary, you can so arrange checkers or blocks or tongue depressors and carry out the counting process. And children would be sure that the same number of checkers could be arranged in 6 rows with 4 in each row. Further, they would realize that "4 columns with 6 in each" and "6 columns with 4 in each" are also descriptions of the arrangement. They would understand that "4 rows and 6 columns" and "6 rows and 4 columns" also refer to the same kinds of arrangements of 24 items of any kind.

This is the first part of the triple play . . . a recognition of a particular pattern in the world of things. Next, this pattern needs a name. A sensible name would be:

$$4 \times 6 = 24$$

This "naming" is the second part of the triple play. It has taken us out of the world of things and we are in the world of abstraction. We have written a series of symbols and have agreed that these symbols point clearly to the pattern we noted in the physical world.

But, let's hurry back to the more familiar world of things. "Think of that name for a pattern as a *headline.* Where might you encounter an example of that pattern?" Or, "Make up a story that fits that headline." Whether the stories are from the world of real things or the child's own world of imagination makes little difference—as long as those worlds operate in such a way that $4 \times 6 = 24$ in the same sense that an array of 4 rows and 6 columns of checkers contains 24 checkers.

"It's 6 blocks to school and I make two round-trips a day, so I walk 24 blocks." This story clearly fits the headline, even though 4 comes in only indirectly—because the child assumes that we understand that four one-way trips and two round-trips are the same distance.

"Twenty-four men went to the moon in four spaceships. Each ship has two pilots and a crew of four. There were six in each spaceship." The last sentence was added, we suppose, for those who want to know where 6 comes into the situation.

"There are 6 gallons of milk in 24 quarts." Does that story fit the headline? Does it point to a pattern that has something in common with that aspect of the array of checkers we named "$4 \times 6 = 24$?" Certainly. Imagine six gallon cartons of milk in a row. Now each carton is replaced by a column of four quart cartons. We now have 4 rows of cartons with 6 in a row—4 rows and 6 columns.

This is the third part of the triple play—finding a few of the unlimited number of "examples" from actual or imagined experiences which display the pattern we are discussing.

Thus, the search for examples is an essential part of the developmental work in the classroom. The children are the searchers. They are deepening their understanding of the relationships that exist between the abstract world of mathematics and the world we know through our senses, the physical world. This is a creative experience.

In such activities, "examples" play a fundamental role in the child's growth in comprehension of the meaning of mathematics.

In contrast, contrived "story problems" in text-

books, often used as tests, lead to a distortion of the relationship between mathematics and the real world—it leads children to say, "My mother might send me to the store for 24 ÷ 4 eggs."

Math Workshop work texts contain very few typical "word problems." However, in the teacher's guides, ample suggestions are given to help teachers stimulate all children in fruitful searches for examples of the patterns of arithmetic the children are considering.

Headlines and Fact Teams

We can think of no useful end served by arguing that one member of a fact team is a more appropriate "headline" for an example than are the other members of the team. Consider this team:

$4 \times 6 = 24$ $6 \times 4 = 24$ $24 \div 6 = 4$ $24 \div 4 = 6$

Our main emphasis here is that all these members are appropriate names for a pattern we see repeated over and over again in the real world. The focus is on the pattern, not on the particular expression that points to the pattern.

Arrange 24 checkers in 6 rows and 4 columns. That arrangement of things has something about it worth noticing. Any member of the fact team points to the pattern. No team member has any claim to being *most* appropriate.

The headlines, $24 \div 4 = 6$ and $4 \times 6 = 24$, are simply two different ways of pointing to a particular pattern. Some would argue that, "Six children confronted with the task of dividing 24¢ equally" ought to be analyzed in such fashion that "$24 \div \_\_\_\_ = 6$" is a proper statement of the problem, and "4¢" the proper solution. This centers the attention on discussion about "What is happening? What are the groups? Can you divide cents by people?"

The "understanding" in this approach is directed toward "understanding rules" to justify the argument that $24 \div 4 = 6$ is correct and that $24 \div 6 = 4$ in this instance is wrong. This must be terribly confusing to children and teachers—and parents.

Shift the concern to the pattern involved. What pattern must you recognize to help divide 24¢ equally among 6 children? The pattern we need to see is the same pattern we saw in the checkers or milk cartons. We named it with a fact team and each member of the team simply calls to mind the pattern!

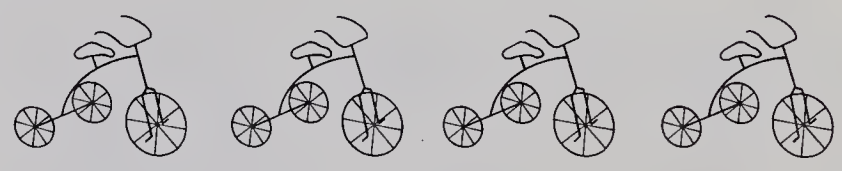
Suppose a child argues: "Half the children will get half the money—or 12¢ for 3 children. So, each child will get a third of 12¢—or 4¢ for each." So he suggests the headline "$(24 \div 2) \times 1/3 = 4$" as appropriate. Is he wrong? We hope not! . . . We hope that $(24 \div 2) \times 1/3 = 24 \div 6$.

Our concern is that children understand patterns and the relationships between the patterns of mathematics and the patterns of the world of experience. We hope they do not need rules to help them conclude that "each child gets 4¢."

Put the following incomplete headlines on the board. Ask the children to complete each so that it is a true statement and then make up a story that "fits the headline" in each case.

A.	$4 - 2 = \_\_\_\_$	B.	$3 \times 4 = \_\_\_\_$
C.	$\_\_\_\_ = 2 + 3$	D.	$12 = 4 + 4 + 4$
E.	$4 \times \_\_\_\_ = 12$	F.	$15 - 3 = \_\_\_\_$
G.	$\_\_\_\_ + 3 = 9$	H.	$\_\_\_\_ \div 4 = 3$
I.	$12 \div \_\_\_\_ = 4$	J.	$8 + 4 = \_\_\_\_$

Several of the headlines seem to point to the same pattern. Which ones? (Candidates will be B, D, E, H, I; and possibly an argument will be advanced for F.) How many fit the following sketch if we direct our attention to "wheels?"



"Finish all the proposed headlines."

$3 \times 4 = 12$ $4 \times 3 = 12$ $12 \quad 3 = 4$

$12 = 4 + 4 + 4$ $12 \quad 4 = 3$ $8 + 4 = 12$

Shall we accept all these headlines as appropriate? Does each point to a pattern that fits the real-life situation we are confronting? The easy way out would be to accept them all as different ways of saying something about the "number of wheels" in this case. That's the easy way . . . and why not take it?

Especially since the alternative has a gloomy outlook to it. We might soon be involved in a philosophical argument that would find us arguing that 3×4 does not equal 4×3. At least, we would have to spin a whole set of confusing rules so that, in light of those rules, we could say that one candidate's headline complies more fully than another. The result of such a contrived obstacle is to make "problem solving" a distasteful, complicated, lifeless effort to remember rules and recognize cue words that will help us recall from memory the right rules so we can come to a correct answer.

Any of the headlines we are considering is but another way of pointing to the pattern of the "number of wheels." As soon as a child has completed any one of the headlines that he thinks appropriate, he is in command of a fact of arithmetic that applies. How does he apply it? . . . if by "rules," he is apt to make a mistake; if by thinking about the situation, he will make a sensible reply.

But this part of the activity—recognizing that there are 12 wheels on the 4 tricycles—is only the initiating and less important part of a developmental situation.

"Now," we say, "think up other examples or stories that fit all those headlines." This leads us toward a creative activity in which children explore the relationship between arithmetic and the real world.

"Three chairs have 12 legs."

"There are 12 legs on 4 three-legged stools."

"In 3 rows of trees with 4 trees in a row, there are 12 trees."

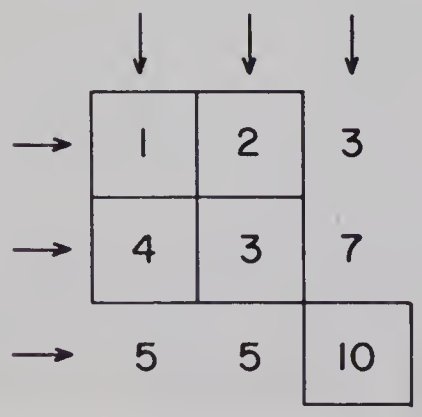
"On 3 wagons, there are 12 wheels."

"Two round-trips to Grandmother's is 12 miles, so she must live 3 miles away."

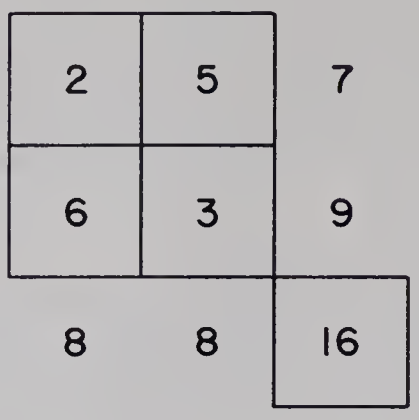
There is a common aspect about all these more or less seemingly diverse situations drawn from the real world. Any of the headlines above point to that common aspect. Any of the completed headlines contains the arithmetic each statement or example relies on.

We ought always to simplify the relationship between arithmetic and the physical world around us. We ought to worry when anyone tries to complicate and propose endless, useless, dangerous rules.

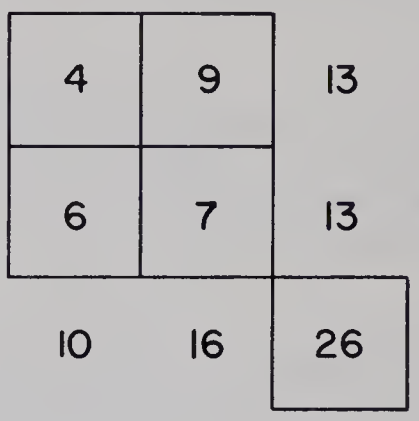
Below is a Cross Number Puzzle form. The problem is to select a number for each of the 4 cells so that if we add across each of the 2 rows and then add their sums, the grand total can serve equally well as the grand sum if we go on to add each of the two columns and total our results. Here is one group of 4 numbers that meets the test:



Perhaps you notice that the sum of each column is 5. Try another group of 4 numbers with the same pattern in the two columns—the sum is 8 in each.



Try an example in which the columns do not follow the pattern we notice:

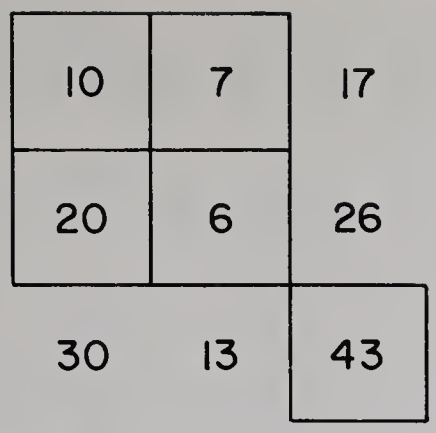


Now the rows add up to the same sum. There's a good bit of symmetry evident and some feel quite confident that this is a scheme that will keep on producing solutions to this puzzle. And it will. But the whole truth is much more startling.

We have explored this idea in the abstract manipulating numbers, inspecting sums, and sums of sums. But, bring it down to the world of things—cows, this time?

Farmer Brown put four cows in one field, nine in another, six in another and seven in the last field. In the two north fields, there are 13 cows and in the two south fields, there are 13 cows—or 26 in all.

Wouldn't it be surprising if there hadn't been the same number of cows regardless of the route over which he drove them? If we came out with any other result, we would think something unnatural had happened along the way. So, we are led to suspect that our puzzle will "work" regardless of what number of cows Mr. Brown puts in those 4 fields.



How clear the whole matter becomes when we take it out of the abstract and find a suitably similar example in the world of objects.

Incidentally, do you notice how quietly an example of "regrouping" and "carrying" was insinuated? If we want to add 17 cows and 26 cows, separate them into fields so that in one column of fields the numbers of cows are multiples of 10.

Headlines and "What's My Rule?"

Now, let's play "What's My Rule?" At first, teacher is "it." She asks the child to call out a number, such as 3. She then "applies her rule" and answers with "6." This example is recorded:

$3 \rightarrow 6$

As children call out numbers, and the teacher applies her rule, results accumulate:

$3 \rightarrow 6$
$5 \rightarrow 10$
$4 \rightarrow$

"I know what you are going to answer," Mary blurts out. "You are going to write 8."

Mary takes over from the teacher. The children keep calling out numbers, and Mary "applies the rule" in responding.

$4 \rightarrow 8$
$1 \rightarrow 2$
$2 \rightarrow$

"You're going to answer by saying 4!" By this time, many of the children think they know the rule.

"Now, let's use these results as a headline. What experience could we be talking about? We could be telling you how far we live from school, and then you tell us how many blocks in a round trip. If I say 2 because I live two blocks from school, you'll answer 4 because that's how far I would walk to school and back. Or maybe Mary and Dick put their books in two piles with the same number in each. The first number tells how big the piles were, and then you tell how many books they started with."

Or, "My father told me he would give me as much as I save from my allowance each week. If I save 4¢, he gives me 4¢ extra and I have 8¢ to put in the bank." And so forth, until the teacher tells the class she has another rule. The list of results accumulates like this:

$1 \rightarrow 8$
$5 \rightarrow 10$
$3 \rightarrow 8$
$0 \rightarrow$

"Five," Bill says firmly.

"If I say 2, you're going to say 7," someone else ventures.

When most children have indicated their grasp of the rule, the results become a headline. What story could fit that headline?

"A boy found a nickel. The first number tells how much he had to start with and the other number tells how much he had after he found the nickel."

"After we drove a ways, we had five miles left to go. If we had gone one mile, then we have five more to go and the whole trip is six miles."

In another "What's My Rule?" game, the teacher asks for a pair of numbers. She rejects some pairs because they don't fit her rule. The following headline is the result:

$3,1 \rightarrow 2$
$5,7 \rightarrow 6$
$6,2 \rightarrow 4$
$0,6 \rightarrow 3$
$4,2 \rightarrow$

"You're going to write 3 . . . and I know a story. "Bill had six books and his sister had two. They put them together and then divided them evenly—each got four."

"In my closet there were three shoes on the shelf and one on the floor. You know now that I have two pairs of shoes."

Another game produces this headline:

$5,7 \rightarrow 5$
$10,9 \rightarrow 9$
$8,5 \rightarrow 5$
$8,6 \rightarrow$

"You are going to answer '6' . . . and your rule is like this. Popsicles are 8¢ at one store and 6¢ at another. You can be sure I only spent 6¢."

So the game moves constantly back and forth

between abstractions of arithmetic and similar patterns in real-life situations. Before we leave each example, every child in the room (almost without exception) discovers the secret rule for himself, if not in the list of abstract numbers, then in the examples from life.

These simple games that involve all children included the idea of multiplication, addition, division, average and inequalities. At no time is any child asked to *verbalize* the "rule." If he could anticipate the result while the headline was being constructed, he was operating on a higher level of discovery than those who saw what was going on only as the examples from experience developed. In either case, the children were moving back and forth between the numbers of arithmetic and their use in the real world.

It is obvious that this game can deal with an unlimited variety of relationships on any level of maturity. Each child is challenged to participate on the highest level for which he is equipped by his own talent and experience.

The headline writer sometimes invents shorthand that seems strange, at first. The story writer needs some clues. Here is such a headline, and a group of hints:

1,2	hints:	0,2	0,3	1,3	0,4	0,3
+0,3		+0,1	+0,2	+0,1	+1,1	0,3
,		0,3	1,0	1,4	2,0	,

Are there enough hints? Suppose there aren't, and the teacher must tell about the boy who first held up all the fingers on one hand and two on the other. Then he raised three more fingers. Now he held up all the fingers on two hands. So we complete the headline this way:

$$\begin{array}{r} 1,2 \\ +0,3 \\ \hline 2,0 \end{array}$$

Can you tell a version of the story that fits this headline? Probably, but again suppose we haven't much help from the class. So we say, "A girl had one nickel and two pennies. She earned three more pennies and wanted to change them so she had the fewest coins. She had one nickel and five pennies. She changed the five pennies into a nickel and then had only two coins—both nickels." The light begins to dawn. Let's try one of the hints.

$$\begin{array}{r} 0,3 \\ +0,2 \\ \hline 1,0 \end{array}$$

"Bill had three pennies in one pocket and two in another. At the bank he changed them for a nickel and had no pennies left."

"I had three fingers up and then put up two more. Then I had one handful of fingers up." But, someone argues, maybe he had three on one hand and two on the other. To fit the headline, we need a rule of some kind—like the "fewest coins" rule—perhaps we require that you must always fill out a handful of fingers if you can. Then the incomplete hint:

$$\begin{array}{r} 0,3 \\ +0,3 \\ \hline , \end{array}$$

Complete this headline and give an example that fits it.

"Three pennies and three pennies equals six pennies. Five of them can be changed to one nickel and there is one penny left or one nickel and one penny in all," (The headline 0,3 + 0,3 = 1, 1 can be read "zero comma three plus zero comma three equals one comma one.")

"We have three school days this week and three school days next week. That's the same as one week of school days and one more day."

$$\begin{array}{r} 1,0 \\ -0,3 \\ \hline , \end{array}$$

"I had a nickel and spent 3¢. That left me with no nickel and two pennies."

"I put down three out of a handful of fingers. There were only two left up."

$$\begin{array}{r} 2,2 \\ -\ 1,4 \\ \hline 0,3 \end{array}$$

A completed headline often stimulates a greater variety of responses—if such are encouraged.

"I have one nickel and four pennies left after spending 3¢. I started out with two nickels and two pennies."

"I can't put up three more fingers if I already have one handful up and four up on the other hand. I would be two fingers short."

$$\begin{array}{r} 1,2 \\ \times\ \ 3 \\ \hline , \end{array}$$

"Each of three girls had one nickel and two pennies. They put their money together and had three nickels and six pennies. They changed five of the pennies for a nickel and then had four nickels and one penny." And, eventually:

$$3\overline{)4{,}1} \quad 1{,}2$$

"Bill had three nickels and a penny. He spent a third of it. That is, one nickel and two pennies. Four nickels and one penny is the same as three nickels and six pennies. One third of each is one nickel and two pennies."

When the children have the idea worked out, a new headline is considered and obviously requires some new hints.

0,8		0,5	0,10	1,5	1,6
+ 0,4	hints:	+ 0,6	+ 0, 4	+ 0,5	+ 1,6
1,0		0,11	1, 2	1,10	

"There were eight eggs in one carton and four eggs in another. All could be put in one carton of one dozen—and none left over."

"The little hand of the clock has gone around to 8. In four more hours it will have made exactly one trip around the clock."

"A board 8 inches long and one 4 inches long can be cut out of a 1-foot board."

"My last birthday was eight months ago. My next birthday will be in four more months because eight months and four months are one year and no months."

Then there is the incomplete hint above—

$$\begin{array}{r} 1{,}6 \\ +\ 1{,}6 \\ \hline \end{array}$$

"A dozen and a half plus a dozen and a half is three dozen."

"Out of 3 feet of ribbon, I can cut two pieces, each 1 foot and 6 inches long."

All these headlines have a bit of a puzzle to them. The notation is strange, but it helps express clearly many events in the real world. Part of the purpose in providing children with such activities is exercise in using what some people call *denominate numbers*—adding 8 inches to 1 foot and 6 inches, and then changing 1 foot and 14 inches into 2 feet and 2 inches. But there is more.

Consider these headlines.

1,7		0,3	0,5	3,6
+ 0,6	hints:	+ 0,7	+ 1,6	+ 1,6
		1,0	2,1	5,2

"Jim had one dime and seven pennies. He earned six more pennies. He changed his dime and thirteen pennies into two dimes and three pennies."

$$\begin{array}{r} 3{,}2 \\ -\ 1{,}5 \\ \hline \end{array}$$

"Mary spent a dime for a candy bar and 5¢ for gum. She started out with three dimes and two pennies. She had one dime and seven pennies left."

$$\begin{array}{r} 1{,}4 \\ \times\ \ 3 \\ \hline \end{array}$$

"If each of three boys took a dime and four pennies to the store to make a purchase, they had three dimes and twelve pennies. First they changed their money into four dimes and two pennies."

One of the important dimensions to this activity is now apparent. In this final series of examples, we find that we no longer need the commas in the notation.

$$\begin{array}{r} 23 \\ -\ \ 8 \\ \hline \end{array}$$

"I had 23¢ and spent 8¢. I had one dime and five cents left, or 15¢."

On our somewhat circuitous route through hands and dozens and dimes, we have been exploring the idea of place value; and, we have been carrying and borrowing without staying long at any time in the abstract world of numbers. Every step we took was based solidly on the physical world of things.

This *headline* technique can be used to help develop every idea of arithmetic on every level of development. It's an old idea that is familiar to all teachers. All teachers have, from time to time, asked children to provide examples in which facts of arithmetic are reflected in life. But, this activity ought to become a theme permeating daily activity.

Examples called "story problems" have usually been reserved for use as tests of what children remember about trips through the abstract reaches of number manipulation. "Now that the class has learned a new group of division facts," we say, "let's see if they can use this new knowledge."

Children can be creators of their own examples—answers and all—and some of theirs will be much more difficult than any that could be included in a textbook. Since most of the creative activity depends on the spoken language, there is less of a reading problem to slow down the arithmetic. Reliance is on the speaking vocabulary which is usually several years ahead of the reading vocabulary.

Children are also given written examples. They must work out the appropriate headline, but that's only the beginning. As soon as they have found and correctly interpreted a headline, they are sent scurrying about for many other examples that would fit the same headline. In this way, usually trivial story problems launch a spirited search for suitable and more personally satisfying examples. Many children demonstrate remarkable ingenuity in creating a fascinating example to fit a headline that started from the question, "How many children are there in two sets of triplets?" You're apt to be out in space in no time at all, or deeply involved in a baseball game, with "two sets of outs" each "inning."

This activity obviously provides adequately for individual differences. There is opportunity for each child to participate on his own level—and to enjoy success.

Each pupil is urged to stretch his efforts in the direction of increased creativity.

Because the acitivities outlined here never lose intimate touch with the physical world, children can move on at an unexpected speed—and with complete understanding. Teachers become the thought-starters, the probers, the leaders who open to children the exciting world of mathematics. Once the interest in pattern and the search for its extension has begun, children are anxious to move on at their own speed. A breath of freedom and sense of adventure enters arithmetic.

If the method proposed here as a main classroom technique has been made to stand on its own feet, then the time has come to give away a secret. The secret was saved for the end—after all the arguments were in—so that skeptics would keep on reading.

The secret is a simple one and one easily demonstrated in any classroom. It is that children actually enjoy looking for their own examples in their own world of fact and fancy. Where this hunt for suitable illustrations has been used consistently in classrooms, it has never failed to excite keen interest in arithmetic. Children have good, old-fashioned fun as they deepen their mathematical understanding.

Other ideas from other levels

Many activities suggested for other levels of the *Math Workshop* are easily adaptable to many levels of maturity. Some are reviewed here as aids to creative classroom work that are rich in mathematical content, stimulating for all, and open-ended to provide for individual differences.

1. The language of addition and multiplication

In arithmetic, there are two processes that make it possible to *operate on* or think about a pair of numbers in order to suggest a particular third number. These basic processes are called addition and multiplication.

Addition or multiplication, and their inverses, subtraction or division, can be discussed in a few meaningful words. The key words used in addition and subtraction are: *two addends and a sum.* Addition and subtraction boil down to a search for a missing term: an addend or a sum, when two of the three terms are given.

	MISSING	GIVEN		
		Addend	Addend	Sum
2+3 = ____	sum	✔	✔	
1+____=8	addend	✔		✔
____+7 = 10	addend		✔	✔
7−3 = ____	addend		✔	✔
7−____ = 2	addend	✔		✔
____−10 = 15	sum	✔	✔	

The key words used in multiplication and division are: *two factors and a product.* Multiplication and division boil down to a search for a missing term: a factor or a product, when two of the three terms are given.

	MISSING	GIVEN		
		Factor	Factor	Product
3×4 = ____	product	✔	✔	
____×7 = 21	factor		✔	✔
5×____ = 30	factor	✔		✔
24÷3 = ____	factor		✔	✔
18÷____ = 9	factor		✔	✔
____÷6 = 7	product		✔	

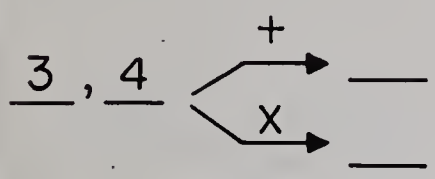
Main credit for this greatly simplified language belongs to the School Mathematics Study Group (SMSG). Instead of preserving the confusing vocabu-

lary that has been traditionally repeated in elementary arithmetic textbooks, SMSG has chosen to reflect the way mathematicians actually think and work. Looking at the operations of arithmetic in this way makes it possible for a pupil to have confidence enough to say, *If I can add, of course I can subtract, and if I can multiply, of course I can divide.*

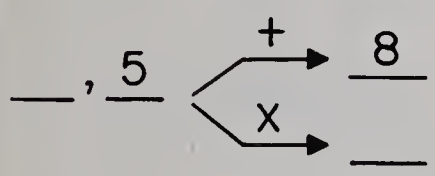
2. Basic fact practice games

When a pair of numbers is considered as addends and as factors that point to a sum and a product, four numbers are involved—the pair, a sum, and a product. *If any two are given, can you find the other two?* (Yes, unless 0 is given as one of the pair and as the product.)

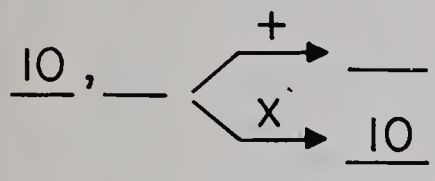
There are many ways to ask these questions. Three suggestions are given below:



(Answer: Sum must be 7 and product must be 12)



(Answer: Missing member of pair is 3 and missing product must be 15)



(Answer: Missing member of pair must be 1 and missing sum must be 11)

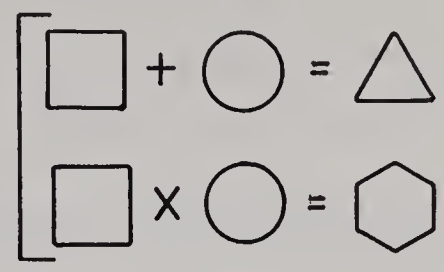
In another form, the questions may be asked this way:

	Product	Pair	Sum
A		3,2	
B	16		10
C	13	1,	
D		3,	6
E	7		

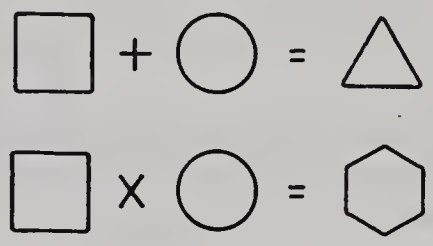
(Missing product is 6 and missing sum is 5)
(Missing pair must be 2,8 or 8,2)
(Missing member of pair must be 13 and sum must be 14)
(Missing member of pair must be 3 and product must be 9)
(Missing pair must be 1,7 or 7,1 and sum must be 8)

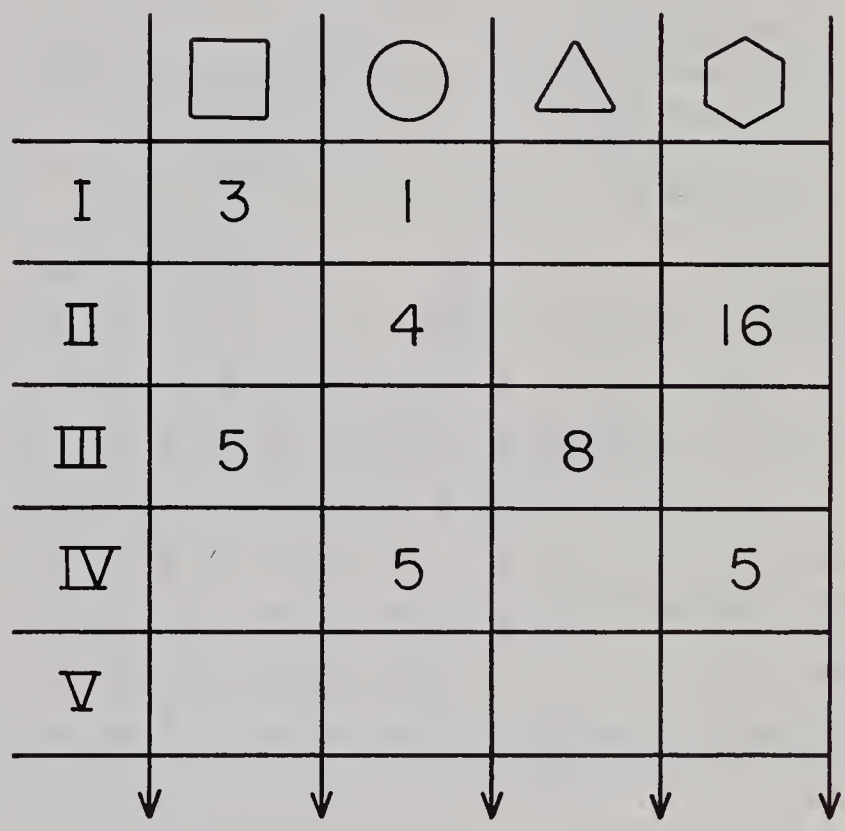
(When the product is a prime number, and it is given, then the pair must be composed of that number and one—the only factors of a prime number.)

In yet another form, the questions may look like this:



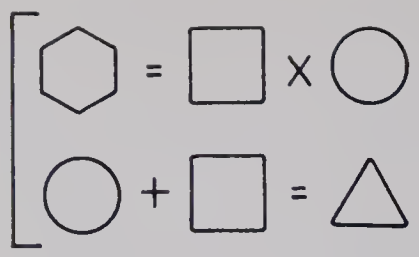
Now, an agreement is in order. In each example, any number that is indicated for a shape must occur in all similar shape's. The rules do not prevent the same number occurring in different shapes.



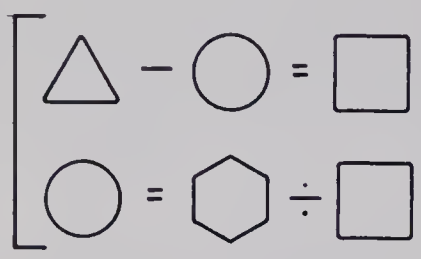


In example I, we are told that 3 and 1 are the pair. Then the sum, or 4, is recorded under the triangle; and the product, or 3, is recorded under the next hexagon. In example II, we are given one member of a pair (4) and the product (16). Since $4 \times 4 = 16$, the missing member of the pair must be 4; then the sum must be 8. In example III, the pair is obviously 5, 3 and the missing product is 15. In example IV, the product is 5 and one member of the pair, taken as a factor, is 5; therefore, the other member of the pair must be 1; and the sum must be 6.

An advantage of this presentation is that the questions may be asked in a variety of ways:



And, later, the same forms can be used in subtraction and division:

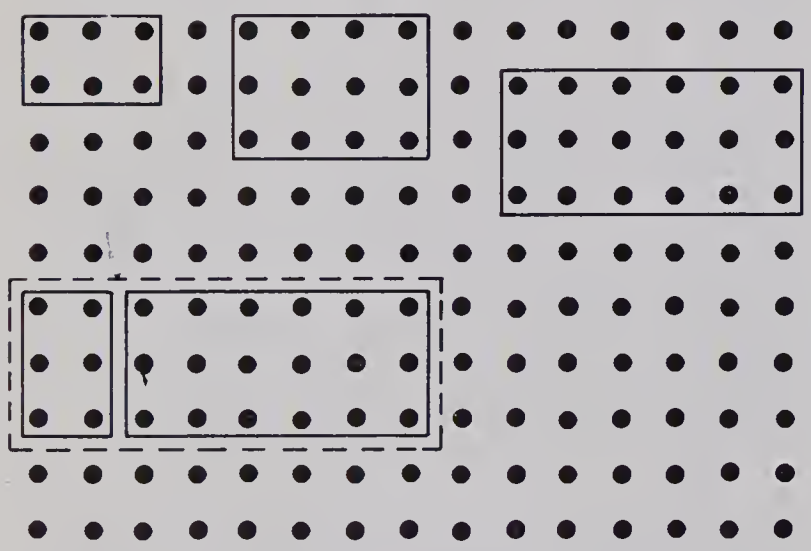


3. Pictures relating addition and multiplication

Begin with a large array of dots on the blackboard. Ask such questions as the following:

"Can someone draw a line around a group of dots to suggest 3×2?"
"Can someone draw a line around a group of dots to suggest 3×4?"
"Can someone draw a line around a group of dots to suggest 3 × (2 + 4)?"

3 x 2 = 6 3 x 4 = 12 3 x (2 + 4) = 18



After several similar examples, put the following three statements on the board:

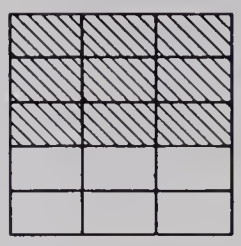
$2 \times 3 =$ ____
$6 \times 3 =$ ____
$(2 + 6) \times 3 =$ ____

"We can use one group of dots to suggest these three sentences. (On the board, mark off the section above that has been outlined with a broken line.) The section I have outlined on the board clearly suggests every part of the long mathematical sentence." Now, let's complete the sentences:

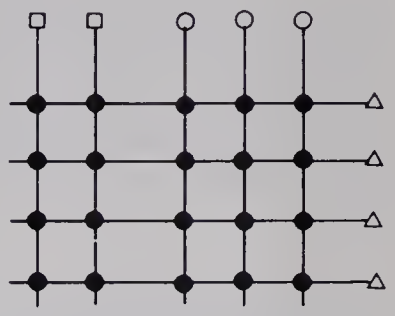
$2 \times 3 = 6$
$6 \times 3 = 18$
$(2+6) \times 3 = 2 \times 3 + 6 \times 3 = 6+18 = 24$

Next, draw several diagrams on the board and ask the children what mathematical sentences each suggests.

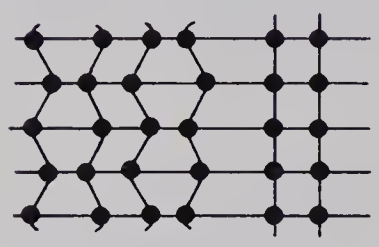
II



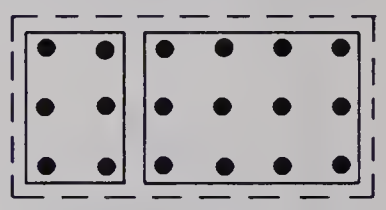
I



VI



III



Mathematical sentences suggested by diagram I would include:

$4 \times 5 =$ ____	$12+8 =$ ____
$5 \times 4 =$ ____	$8+12 =$ ____
$4 \times 3+4 \times 2 =$ ____	$20-8 =$ ____
$20 \div 3 =$ ____	$20-12 =$ ____
$4 = 20 \div$ ____	$(12+8) \div 4 =$ ____

Similar sentences are suggested by each of the other diagrams. After sufficient sentences for each diagram have been noted, reverse the procedure by asking pupils to find the diagram that is suggested by a mathematical statement. "Which of the diagrams (shown above) is suggested by:

$3 \times 5 =$ ____	$5 =$ ____ $\div 6$
$18 \div 3 =$ ____	____ $-6 = 12$
$2 \times 3+3 \times 3 =$ ____	$10+20 =$ ____
____ $-3 \times 4 = 2 \times 3$	$5 \times 6 = 5 \times (2+$ ____ $)$
$6 \times 3 = 6+$ ____	$20-3 \times$ ____ $= 2 \times 4$

Have pupils volunteer to make up sentences suggested by the diagrams. The rest of the class must tell which diagram is being referred to in each sentence.

4. Games of logic

These games can be played many times with the children. With them, they learn to reason mathe-

matically, and then to extend their ability at logic to other areas. Each "If..., then...," statement employs mathematical logic or reasoning. Many of these games are also exercises in precise expression. The games are not fun unless we are sure that everyone says exactly what he means. This gives you, the teacher, an opportunity to emphasize the need, in certain circumstances, for precise expression of ideas.

a. *What's My Rule?* In this game of logic, the leader first asks for a number to which he then applies a rule, such as *add 2; subtract 3;* or *multiply by 10.* Others submit numbers to which the leader applies his rule until someone thinks he knows the rule. He must prove his discovery by giving the second number when another number is submitted. Examples:

Numbers submitted by others–	1	3	10	8
Response of leader–	2	4	11	9

(Rule: *The next number, or one more.)*

Numbers submitted by others–	2	7	5	12
Response of leader–	4	14	10	24

(Rule: *Twice the number.)*

Below are listed some of the rules which might be applied to the numbers 2 and 4 in order to get a third number:

2,4	2	(the *smaller,* or *first,* of the pair)
2,4	6	(their sum)
2,4	8	(their product)
2,4	3	(their average)
2,4	4	(the larger, or second, number)
2,4	5	(the second number and 1)
2,4	24	(the numbers side by side)
2,4	12	(twice their sum)

It is also possible to move out of arithmetic with this form of "What's My Rule?" Put the following on the board. Can the children finish the examples and make up some of their own that follow the same pattern?

1) a→b
b→c
m→n
s→__
__→s
__→r
__→__
__→__

2) a→c
b→d
f→h
r→__
__→x
__→f
__→__
__→__

3) a→A
b→B
c→C
m→__
__→X
__→Y
__→__
__→__

4) c→b
e→d
t→s
d→__
m→__
__→h
__→i
__→__
__→__

5) c→a
h→f
n→l
r→__
q→__
__→__
__→__
__→__
__→__

6) a→B
f→G
r→S
b→__
d→__
__→T
__→Q
__→__
__→__

7) band→and
bone→one
bin→in
bark→_____
_____→owl

8) 23→3
37→7
10→0
99→__
__→8

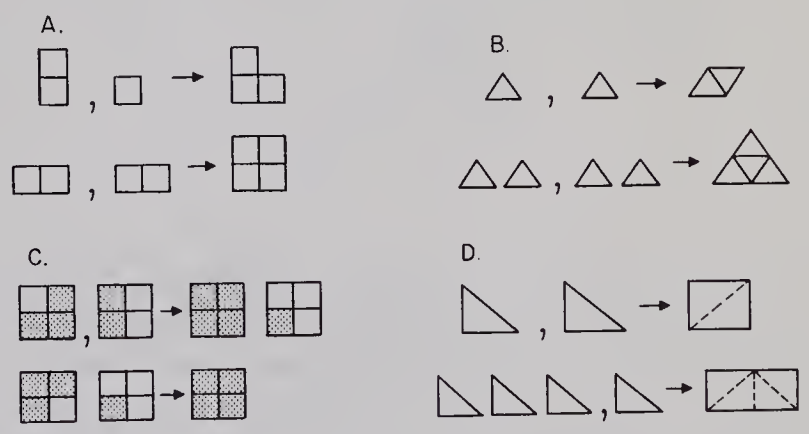
9) man→men
bird→birds
woman→women
mouse→_____
_____→feet

10) (present→past)
give→gave
have→had
hit→hit
run→_____
_____→fought
_____→_____

11) (reverse first and last letters)
now→won
saw→was
rat→tar
tip→____
pet→__
____→____

12) (reverse order)
123→321
37→ 73
8→ 8
25→__
____→34
____→____
____→____

Below are various examples that will help pupils see the relationship between the sizes and shapes of things:



E.
be, at→beat
my, self→myself
of, ten→often

F.
upon, on→up
into, in→to
today, to→day

G.
bother, am→wing
five, two→a
mother, up→year

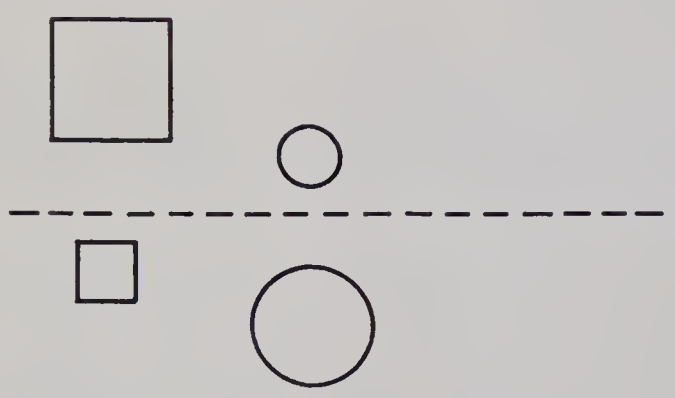
H.
mask, m→ask
tear, r→tea
swing, w→sing

In each of the above cases, a rule or a pattern is suggested by a few examples. As soon as pupils recognize the rule, they can complete other examples according to the rule, and they can make up examples of their own. There is no need to verbalize the rule—simply continue to illustrate it in examples.

The first child to understand the rule can be made an "expert" to test others who think they have discovered it. When two or three children have become experts, the teacher can help those who are still not tuned in, while the experts continue to test others.

Let children make up their own series to suggest a rule they have in mind. They can put their examples on the board, and act as "experts" for those who think they have discovered the secret. In this way, many activities can be going on at once without a teacher's direction, and the teacher can see that everyone succeeds in discovering some patterns.

b. ***Fewest Questions.*** Here is a game that strengthens number concepts, expands vocabulary and develops powers of reasoning. It is called "Fewest Questions." You can start with a very simple example on the blackboard or on a piece of oak tag.



Some preliminary questions might be: "How many shapes are there above the line? How many are there below the line? How many are there altogether? How many are squares? How many are circles? How many are large? How many are small?" Now to the game.

"I'm thinking of one of those shapes. Try to guess which one it is. We'll keep score of how many questions you ask."

Select a child as scorekeeper at the board to make a tally each time the guesser asks a question. Except at the outset, discourage having to point to the diagram in asking questions.

"Is it the big square?"

"No, it's not the big square. That's your first question."

"Is it the big circle?"

"No, it's not the big circle. That's your second question."

"Is it the little circle?"

"No, it's not the little circle. That's your third question. Do you know the shape I'm thinking of?"

Of course, if the guesser remembers the questions, he would know that it has to be the "little square" but he may keep on asking until he gets the answer.

After a few such guessing sequences (in which you may take the liberty of saying *No* as often as you logically can), offer to be the *guesser* yourself.

"I'm going to turn my back. Barney will come up to the board and point to one of the shapes. You will all know the one he selects—but I won't. Then I'm going to find out by asking just two questions!"

Your questions may be:
1) Is it a square?
2) Is it above the line?

There are only four possible combinations of answers. They are

	A	B	C	D
Is it a square?	Y	Y	N	N
Is it above the line?	Y	N	Y	N

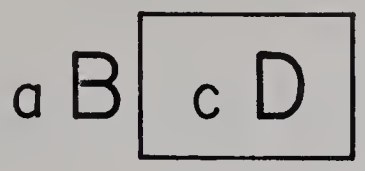
If the answers to both your questions are "Yes" (A)—it is the large square.
If the answers are "Yes" and "No" (B)—it is the little square.
If the answers are "No" and "Yes" (C)—it is the small circle.
If the answers are both "No" (D)—it is the large circle.

You might ask another pair of questions.
1) Is is small?
2) Is it a circle?

Again, any combination of Yes-No answers will reveal the chosen shape. The secret is that each question must divide the group in half in a different way.

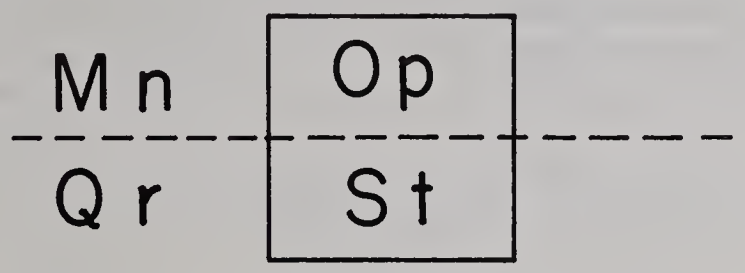
After this, you should return to the children

as *guessers* and *selectors*. (Some will get the idea right away. They've discovered the secret and they must keep it.) Change the diagram.



1) Is it a large letter?
2) Is it in the box?

After the children become adept at asking the questions so they never need more than two, enlarge the diagram.



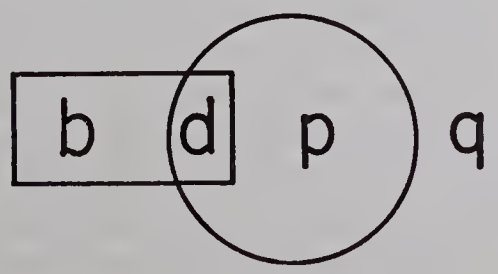
"How many large letters? How many small letters? How many letters altogether? How many letters above the line, how many letters below the line? How many inside the box? How many letters are both above the line and in the box? (Only two—O and P.) How many large letters are both below the line and not in the box? (Only Q.)"

That last question suggests a group of three questions that will reveal any choice:

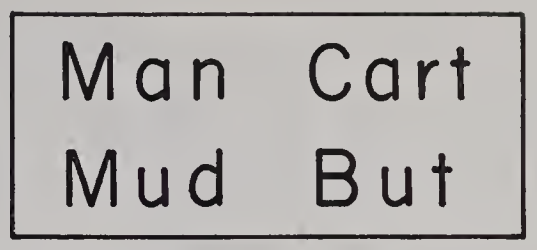
1) Is it a large letter?
2) Is it above the line?
3) Is it in the box?

(1)	Y	N	Y	N	Y	N	Y	N
(2)	Y	Y	N	N	Y	Y	N	N
(3)	Y	Y	Y	Y	N	N	N	N
	O	p	S	t	M	n	Q	r

"Fewest Questions" is a favorite with the children. Let them take their time in discovering the secret. You may vary the objects to include geometric shapes, letters of the alphabet, short words, numberals, etc. Below are a few variations in design with examples of the fewest questions that will expose any item chosen.

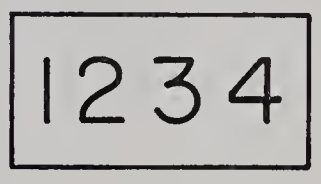


1) Is it in the rectangle?
2) Is it in the circle?

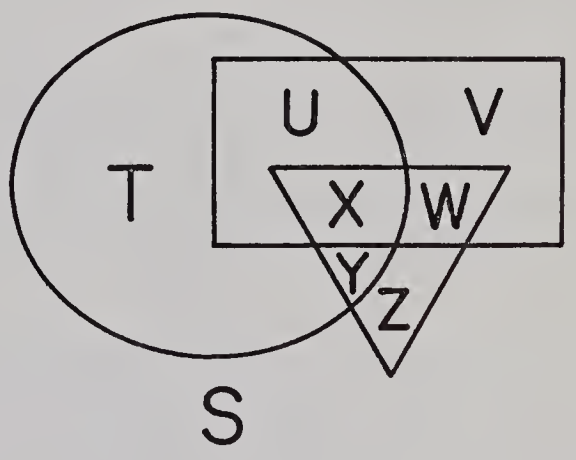


1) Does it begin with *M*?
2) Is there a *u* in the word?

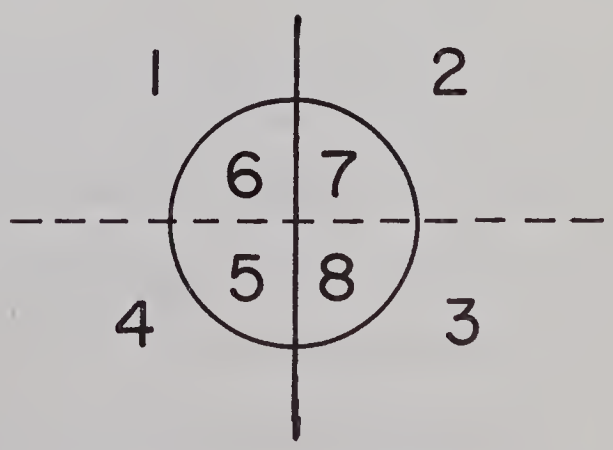
1) Is it in the word BARK?
2) Is it in the word TUB?



1) Is it one or four?
2) Is it more than two?



1) Is it in the circle?
2) Is it in the square?
3) Is it in the triangle?



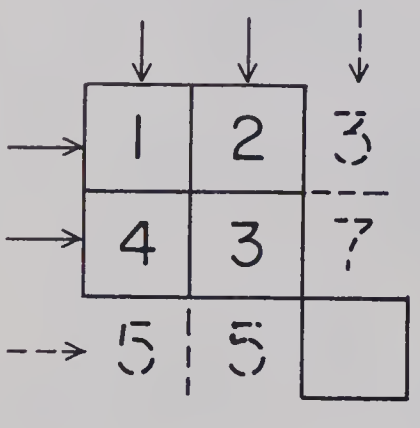
1) Is it above the line of dashes.
2) Is it to the right of the solid straight line?
3) Is it in the circle?

E	T	A	O	I	N	S	H

1) Is it in the word WINTER?
2) Is it in the word FINISH?
3) Is it in the word EASIER?

5. The Cross-Number Puzzle

On the board, draw the following:



The arrows, you will explain, tell us to *add*. As results of the four examples of addition indicated by the solid arrows are recorded, we are led to wonder where to record the additions indicated by the broken-line arrows.

Of course, in one example shown, we could write 10 in the box and it would serve jointly as the result of 3 + 7 and of 5 + 5. Is this a coincidence? Most children and adults think it is, and that this outcome is dependent upon the choice of the four numbers originally written in the box.

Many of your children would be less surprised by the outcome if they imagined objects in the four cells—one dot, two squares, three circles, and four cars. We can sweep the objects across into piles of three and seven; then sweep them down into a single pile of ten. Or, we could sweep them down into piles of five and five; then sweep them across into a single pile of ten.

If we started with ten objects and swept them together we would not expect any to appear out of thin air or to miraculously disappear. If we start with ten objects, we will end up with ten objects.

We might keep track of the goings-on this way:

$$(1+2)+(4+3)=3+7=10$$

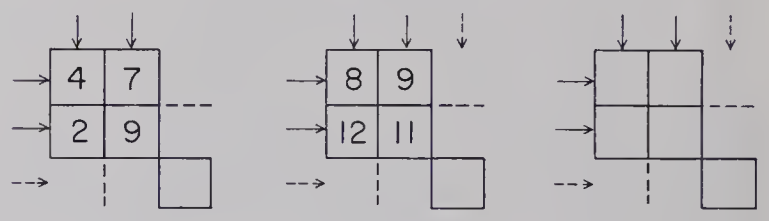
If we had, however, swept down both columns and then across the bottom row, we might keep the following record.

$$(1+4)+(2+3)=5+5=10$$

There is a fundamental rule in arithmetic that the order in which numbers are added does not affect the result. In columns of addition, we often "add up" the columns and then, to check our computations, we "add down" the columns, because we are sure that the order in which we add ought not to change the result.

Yet, when most children and most adults are first confronted with a Cross-Number Puzzle, they are amazed that the box in the lower right-hand corner will suffice for recording the addition in both the right-hand column and the bottom row.

Try the next two examples and then write any four numbers that come to mind in the third:

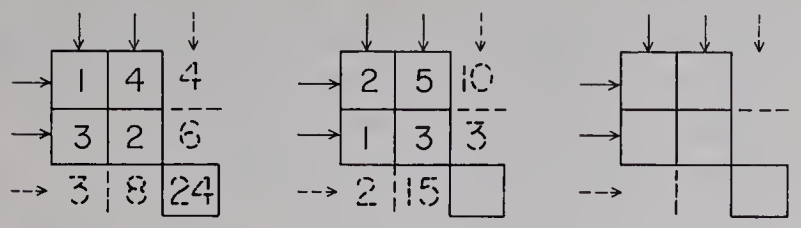


You probably have several children in your class who can give this puzzle a real work-out while trying to discover a group of four numbers that won't work. Even though they have swept objects on the floor and visually swept designs on paper, first one way and then the other, they do not fully comprehend that those numbers in the cells are answers to "How many?" How many what? Anything you like or don't like, or that can be counted.

a. In-any-order rule. In this work, you are helping children see examples of a basic idea in mathematics: *When we add, the order in which we proceed is immaterial.* We have nicknamed this rule the *in-any-order* rule. (Mathematicians will analyze it more carefully and tell us we are combining two fundamental laws, the Associative law for addition and the Commutative law for addition. The *in-any-order* rule for addition seems more descriptive and is easier to remember.)

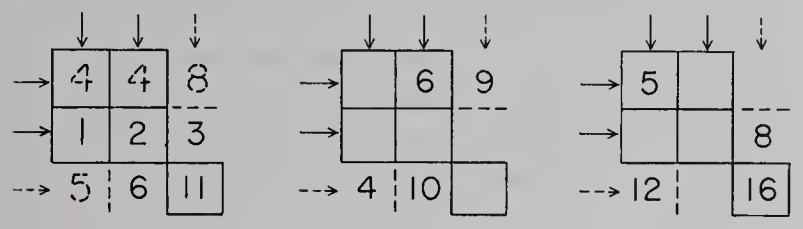
b. In-any-order works in multiplication. This same puzzle helps reveal that there is an *in-any-order* rule for multiplication, too. In the examples below, the arrows tell us to *multiply*. (In the third example, try any four numbers that come to mind.) Children will be helped by this puzzle form to learn what mathematicians call the Commutative law for multiplication and the Associated law for multiplication which we, for the sake of clarity, brevity

and usefulness, combine into the *in-any-order* law for multiplication.



c. Subtraction—the inverse of addition. The Cross-Number Puzzle will also be used to present subtraction as the twin of addition, its opposite or *inverse.* The agreement for the puzzle below is, as earlier, that the arrows indicate addition and that the bottom right total is the result of adding the four numbers in the basic four cells. We are now faced with a problem of reconstruction, or building backwards.

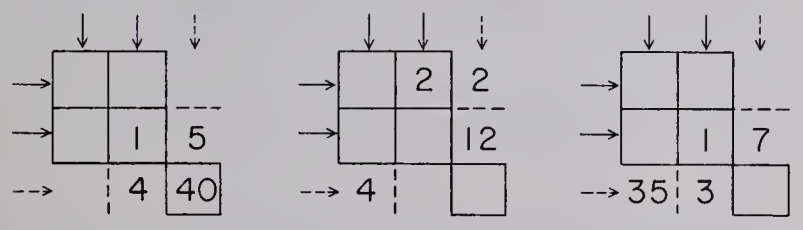
Addition:



To rebuild, in the first example, you may argue: I'm told that ____ + 3 = 11 or in other words 11 − 3 = ____. I know that 8 + 3 = 11 − 3 = 8. Therefore, "8" must go in the top cell on the right column of the first example.

d. Division—the inverse of multiplication. Similarly, we can reconstruct puzzles when the rule is that the arrows indicate multiplication.

Multiplication:



In the first example the question is asked, "What number multiplied by 5 results in 40?" Or, "40 divided by 5 equals what number?" Etc.

The problem of reconstruction now involves *division.* Again, we have a dramatic example of division as the opposite of multiplication.

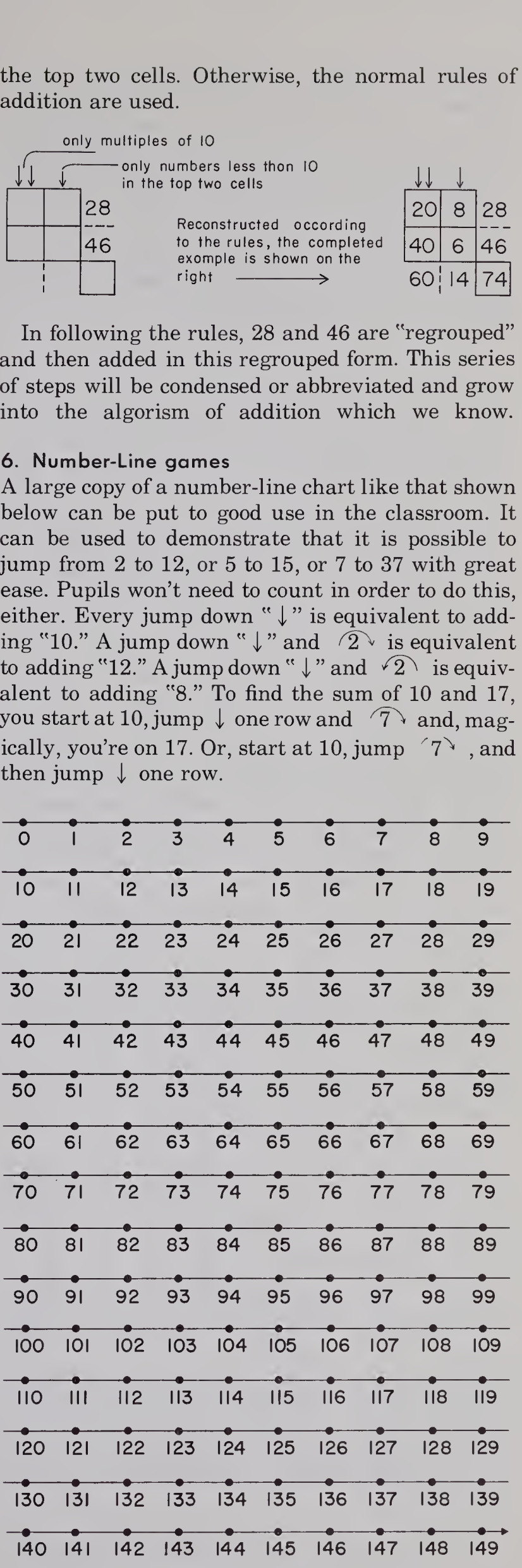
e. Help with algorisims. Cross-Number puzzles help with the algorisms children will later learn to employ when necessary. The examples below call for reconstruction according to the agreement that, in the column under the double arrow, only multiples of 10 can be used and in the column under the single arrow, only numbers less than 10 can be used in the top two cells. Otherwise, the normal rules of addition are used.

only multiples of 10
only numbers less thon 10 in the top two cells
28
46
Reconstructed occording to the rules, the completed exomple is shown on the right
20 8 28
40 6 46
60 14 74

In following the rules, 28 and 46 are "regrouped" and then added in this regrouped form. This series of steps will be condensed or abbreviated and grow into the algorism of addition which we know.

6. Number-Line games

A large copy of a number-line chart like that shown below can be put to good use in the classroom. It can be used to demonstrate that it is possible to jump from 2 to 12, or 5 to 15, or 7 to 37 with great ease. Pupils won't need to count in order to do this, either. Every jump down "↓" is equivalent to adding "10." A jump down "↓" and ↷2↷ is equivalent to adding "12." A jump down "↓" and ↶2↷ is equivalent to adding "8." To find the sum of 10 and 17, you start at 10, jump ↓ one row and ↷7↷ and, magically, you're on 17. Or, start at 10, jump ↷7↷, and then jump ↓ one row.



Allow your pupils to spend several days in learning how to move about the number-line grid. They will enjoy exercises like these: "Where will you land if you start at 0 and take three jumps down? Four jumps down? Two jumps down? Six jumps down? Ten jumps down?

"How many ways can we make the trip from 11 to 22? From 5 to 12? From 3 to 24? From 7 to 18? From 25 to 37? From 111 to 123?

"Where are we if we start on 15, jump down two rows and four jumps to the right? (15 + 20 + 4 = 39) If we start at 15, take four jumps to the right, and jump down two rows? (15 + 4 + 20 = 39) If we take two jumps to the right, jump down two rows, and jump two to the right? (15 + 2 + 20 + 2 = 39)"

Have pupils jump forward or back, and then up, to subtract on the number line. "Just as there are several ways to get from 15 to 39, there are also several ways to get from 39 to 15."

When pupils understand how to use the number-line chart, it is possible to play games with it. For instance, one game is started by saying, "I'm thinking of a number. Start at 15, jump three rows down and four to the left. What is the number?" Whoever can tell you first what the number is becomes the leader and has to think of a number for which he must give appropriate clues. Some pupils may become so proficient that they won't need to use the number-line chart. Encourage such pupils to "do it in their heads."

Other games could proceed like this: "Start at 0, add 11. Add 11 more. Add 11 more. Where are you?" Or, "Start at 3, add 6. Add 3 more. Add 6 more. Where are you?" Use repeated additions and subtractions of 3,4,5,6,7, etc, and ask pupils to look for patterns as they progress.

Jumping on the number line can be used to help underline the fact that in subtraction we simply *undo* what we *do* in addition. If *jump down one row* is equivalent to adding 10, then *jump up one row* must be equivalent to *subtracting 10.* If a jump of *two to the right* is equivalent to *adding* 2, then a jump of *two to the left* is equivalent to *subtracting* 2.

a. Regrouping numbers. In order for a pupil to add 8, it is possible to add 10 and subtract 2, or to subtract 2 and add 10. Confronted with 17 + 8, pupils can find the sum by regrouping the 8 in such a way that the sentence becomes: 17 + 10 − 2, or 17 − 2 + 10.

It is not possible to outline a set of rules for regrouping numbers. As each pupil gains confidence in putting the idea to use, he will also gain an appreciation of when regrouping is especially helpful to him in simplifying a problem. As they jump back and forth on the number line chunks to add and subtract, children learn to feel what's going on, because they *see* what they are doing.

To subtract 3 from 61 on the continuous number line is simply a matter of taking a jump of 3 backwards, but on the number-line chart 61−3 must be handled differently. The problem has to be approached as 61−10 + 7, or 61 + 7 − 10.

b. The decimal system and shortcuts. The structure of the decimal system is emphasized in the way the number-line chart form is designed. Bases for understanding the algorisms of adding and subtracting can be firmly established by developing the ability to jump freely on the chart. Enterprising children will discover many shortcuts for making jumps. The search for shortcuts should be encouraged, because in discovering shortcuts, pupils also discover basic patterns in the decimal system.

7. Discovering matching properties

Research on *discovery in arithmetic* has revealed important information. When children discover a relationship, they are excited and anxious to stay with that discovery. They will eagerly put this new idea to work, showing more examples of *it.* They come alive, often with an "Oo-o-o! I've got it!" or with a burst of industry using a newly discovered relationship to reel off answers with confidence.

What is the *it* they have glimpsed? Well, if we distract their attention with a request for them to *tell about it* or *put it in words,* we have selected a method that could not be better calculated to mutilate the experience.

That a child can move ahead, putting *it* to work successfully is certainly all the evidence we need to know he *has it.* Our only justification for veering off in the direction of a semantic exercise is a feeling that *it* is a generalization that ought to be verbalized. But research does not support this as a valid reason.

Transfer to future experiences is not increased by *verbalizing a generalization.* Sub-vocal awareness, the new-found ability to move ahead more speedily, more directly, more confidently, the discovery of a new and important relationship, will stay around as an effective tool even though it hasn't been named and talked about. An opportunity to put this tool to work will contribute most to its staying power.

8. Silent teaching and discovery

The silent technique of instruction has revealed the mechanics of discovery unencumbered by any attempt at verbalization. A class can be kept on the edge of their seats for an uncommonly long time (and, incidentally, almost alarmingly quiet) as you move on from one level of difficulty and involvment to another. Though the way leads from the familiar deep into new, unfamiliar territory, interest is high, reactions quick, and with the teacher's eyes moving to the fingers, ordinarily timid children can be drawn into participation. The most retiring or placid child may, all at once, seem to come alive. Children who usually are ready first with the answers, are not as effective in keeping in the center of the spotlight.

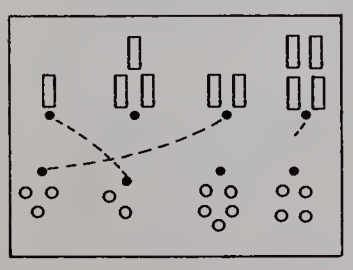
At the conclusion of a silent session, there is really nothing to talk about. And no one seems interested in talking about the experience. But as soon as you lose your voice again, there is an immediate awakening of the feeling of excitement. If a similar activity is initiated several days later, it will move faster than before. No generalizations were verbalized, yet something of what happened has been transferred from the trips into silence.

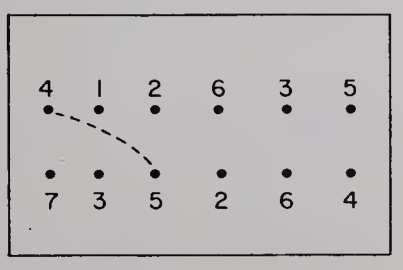
The only danger involved with arithmetic without words is that of over-stimulation. *Discovery* is not a relaxing state, physically or mentally. An act or moment of discovery is a moment of elation that seems to release a dynamics that can be almost self-consuming. The look of eagerness on so many faces all at once is infectious and it is tempting to see how fast and how far you can go.

The best way to assure a moderate forward movement is to concentrate on participation by everyone. If it's a board activity, everyone should handle the chalk. If it's a manipulation of blocks on a table, everyone can show he knows the secret.

One rule of teaching that will multiply the opportunities for discovery in any activity, arithmetic included, is to use the fewest possible words. And when you must talk, leading questions are much more fraught with educative value than any explanations you can make.

a. The game of "Next After." Without any words, put the following examples on the board:





The rule you have in mind is, of course: *One more,* or *the next label on the number line in a direction away from zero.*

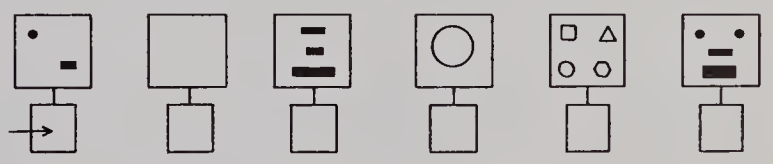
Complete the first connecting line, one square to two circles. Begin the next line and stop, looking for volunteers. Many children may think you have made a mistake. You must show confidence in your connecting. Some volunteers may make unsuccessful guesses as to your secret rule. If no one guesses it, you may have to draw the connection between the dots associated with two squares and three circles.

Start a third connecting line from the dot under the four squares. At worst, a child will connect it to four circles. You are troubled. You retrace the first and second connecting lines. With the chalk, you touch the single square and then the two circles. You express approval. Then you touch the two squares and the three circles. Again you are pleased. And again you look for volunteers. You may even have to complete the entire first example by yourself, though this would be unusual.

Move on to the second example. If no one has read your mind in the first example, begin by connecting the dot associated with four in the top group and five in the bottom group. Proceed as in the first example. You will be surprised by the reaction of the children. This time they will get the idea.

Why? Experiments lead to the conclusion that children understand the "Next After" *numerical game* quickly when it is associated with a memorized sequence of words or symbols rather than in the context of a group that has one more member than another. If you encounter this development, place increased emphasis in counting collections of things and arranging them in order from smallest to largest and from largest to smallest.

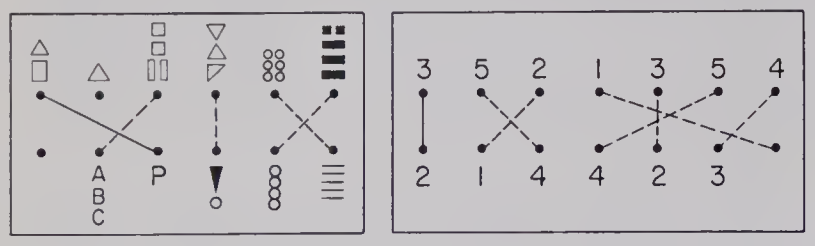
Erase the two examples and go back to labeling pictures of groups.



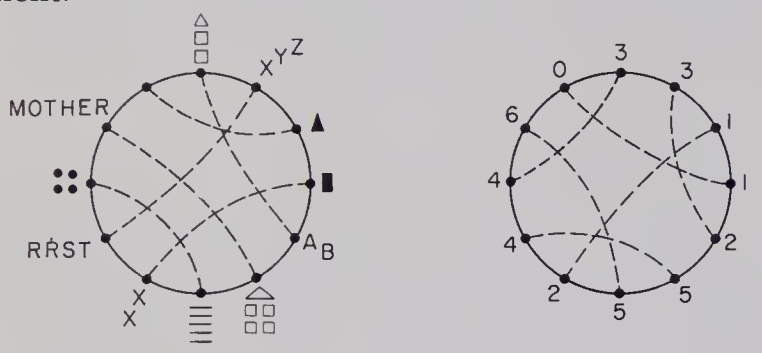
Then go on to another series of pictures. Skip the first, put one design in the second, two in the third, and stop just as you are about to start the fourth. Any volunteers, you seem to ask?

The idea is, of course, the same as with the original pair of examples: *one more,* in another context.

b. More silent teaching. If you feel that your class's response is advanced enough, go on to more advanced silent treatment examples. Use pairs of examples, one with designs, the other numerals.

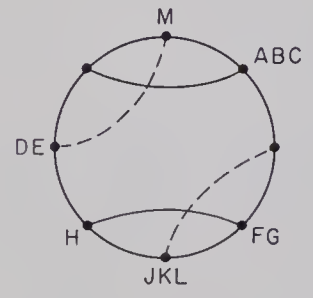


Starting at a dot associated with the pictures in the top row, draw a line to a dot above a group with one less in the bottom row. (Note that *0* or *none* is *one less than one.*) Then move to a circular arrangement:

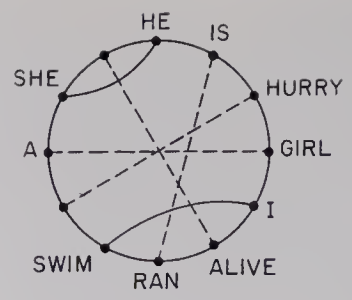


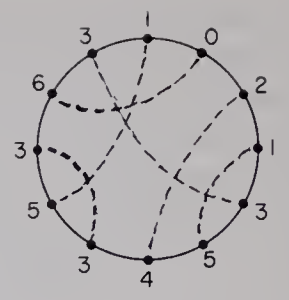
Now, one more and one less are two different ways of looking at a line. There is no general pattern of *from top to bottom* as in the first examples. Further, there are alternate ways of completing the connections. Both illustrations are quite similar. The second can be seen as recording the answers to *how many* designs at the corresponding point of the first. But, the connections are drawn to illustrate how different they may be in the two cases.

Jump to another rule. It also has a name, but there need be no hurry to label it.



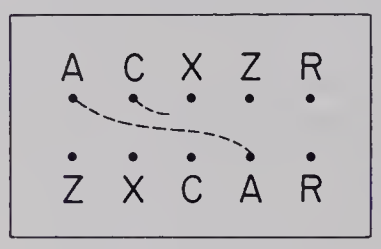
Make two connections, then look for volunteers. This is far from obvious, but it is one of the most important ideas in arithmetic. Our hope is that children will be able to read your mind and make right connections many times and in many contexts before we name the idea *addition.* Try an extension of the idea:

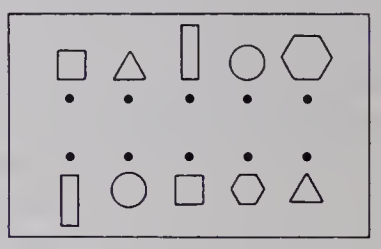




The secret in the first example is *five*; in the second, *six.* Some children may be thinking along these lines, but it would spoil the experience to talk about it. They are reading your mind. They glimpse a pattern that must parallel yours because they can help you draw the connections your patterns would dictate.

c. Matching designs. You have laryngitis again. Put examples such as the following on the board:

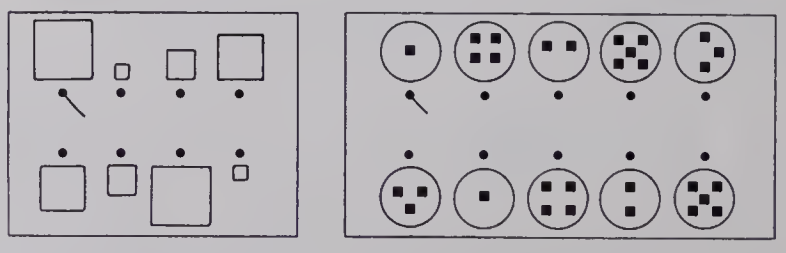




Start drawing a line from the dot under the "A" and stop. You don't have to do this job you were about to do. Look around for volunteers. There are probably several already. Hold out the chalk to one. If he continues the line you started with until it connects to the dot over "A" in the bottom row, you beam from ear to ear because your mental telepathy was successful.

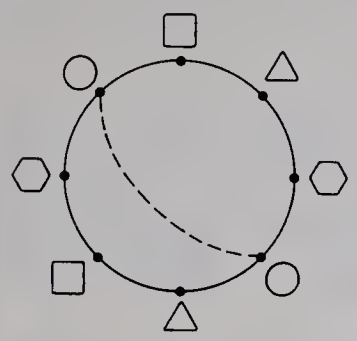
If the volunteer does anything else, you are upset because your telepathic effort wasn't well enough carried out. Remember, you have not said a word, no one knows the rules for sure, no one can make a mistake. You have to try again, and again until you have *communicated.*

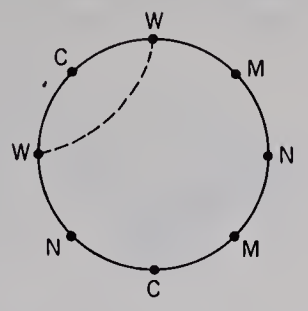
Start a line from the dot under the "C" in the top row. Again decide against extending that effort and look for volunteers. There will probably be many by now and you can move rapidly through both examples of matching. Next, try examples such as these:



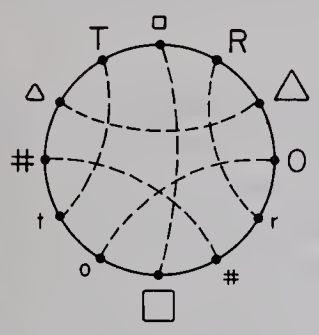
A similar activity, a bit more sophisticated,

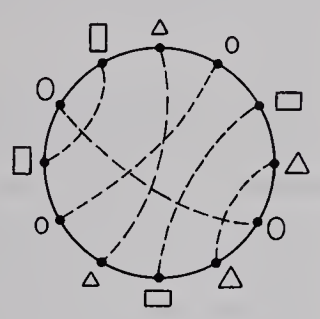
develops when the objects or pictures to be matched are not displayed in two rows but, rather, around a circle.



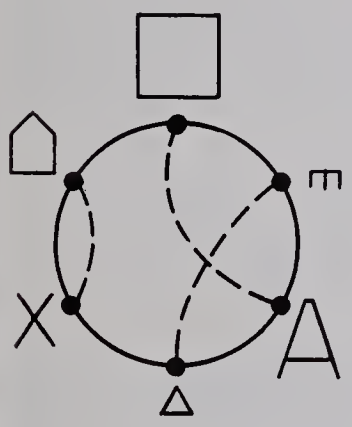


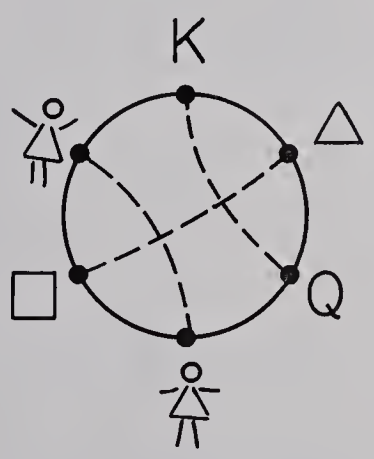
Then more matchings and more sophistication. (In the second example below, there are four of each kind of symbol, so the connections can be made in several ways, any one of which is as good as any other.)



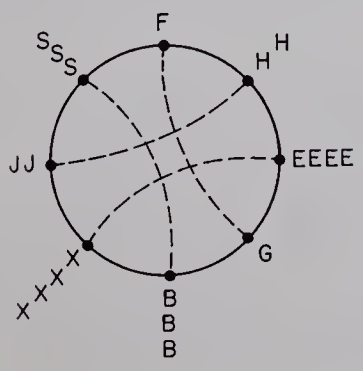


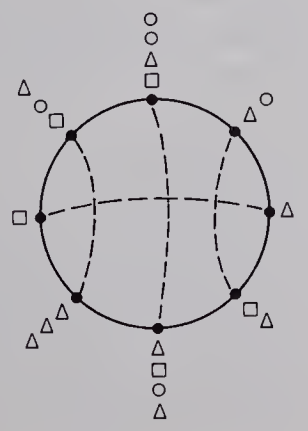
d. Largest, smallest, in-between. You still have acute laryngitis. In the first example below, largest, smallest and in-between, are the basis of matching. In the second, letters, shapes and people.



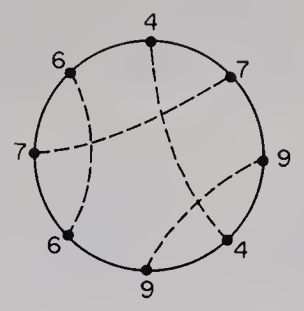


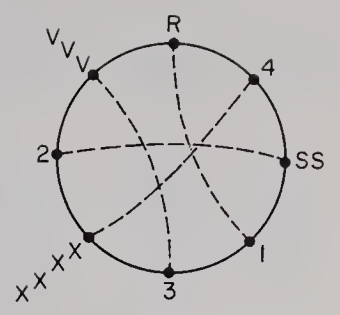
More examples:



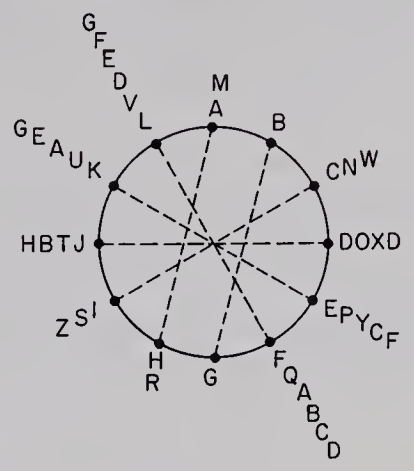


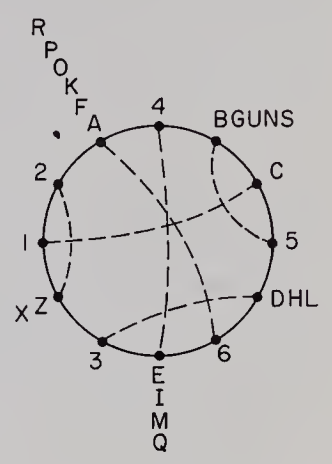
And on to the following:





You still haven't said a word, but activity is going ahead rapidly.





We have come a long way in a short time, moving through several very different kinds of matching, changing rules as we went, all without a word. This technique was first developed as a suggested answer to the question: "How could you teach arithmetic to young children if you did not speak nor understand their language?" Experiments led to the conclusion that you might find a real advantage in not being able to let the confusion of words distort the pictures.

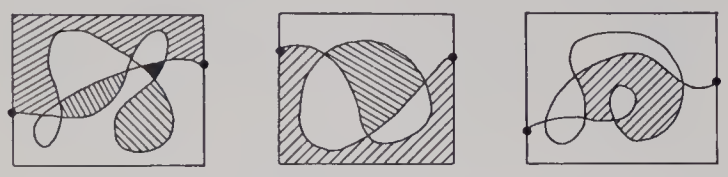
Number is an abstract idea. It is a *property* that groups of objects or other collections may have in common. If they do have this property in common, we say they "have the same number" and we assign a name and a symbol that serves to point to that particular *number* or *property that is common to a whole collection of groups.*

The sum total of the preceding paragraph illustrates what happens when we try to talk about the idea we were freely communicating to children before we resorted to *words.* Every matching was based on recognizing *common properties.* We ended with a matching between collections of letters and the symbol indicating their *number,* indicating the collection of collections to which they belong when number is the property we are noting.

9. The study of regions

Many experiences usually classified as art or "free expression" can be given a mathematical

orientation. For example, draw several rectangles on the board. Ask children to draw a continuous line from one side of the rectangle to the other, letting the line wander about the space. You may have results such as this:



Now, we must define a space or a region. Suppose the frame of the first rectangle is a fence and the lines we drew are fences, too. Imagine a cow eating grass. The cow is located at *a*. Color the area where there is grass the cow can eat.

In the next rectangle, imagine there are two cows. They are in different pastures. Where might they be? One might be at *b*. Color it in. (This area is chosen to demonstrate that the outside of the rectangle is also considered as a fence.) Where might the other cow be? Have children point to the area, and color it in.

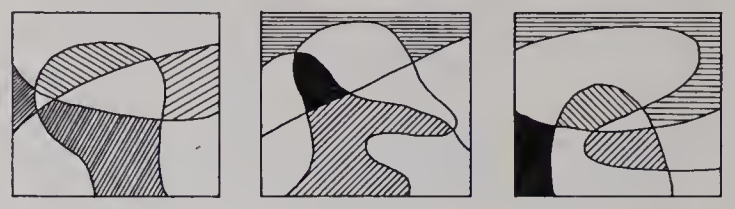
In the third rectangle, imagine there are four cows, each in a different pasture. Here are four different colored pieces of chalk. Use a different color for each pasture. (Four different children can work on this example, one after the other.)

Next, to individual work. Have everyone draw a wandering fence with a black crayon on a piece of paper. Have them draw another wandering fence in black crayon on the reverse side of their papers. On the first side of the paper, they are to imagine there are three cows, each in a different pasture. On the reverse side, they are to imagine there are four cows, each in a different pasture. They are then to color in the pastures where they imagine the cows to be.

The number of cows can, of course, be varied at will. After the activity has been tried once or twice, your directions may change. Two cows are in red pastures and three cows are in blue pastures. With a red and a blue crayon, the children should color in those pastures. Write a summary of instructions on the board:

2 red pastures
3 blue pastures

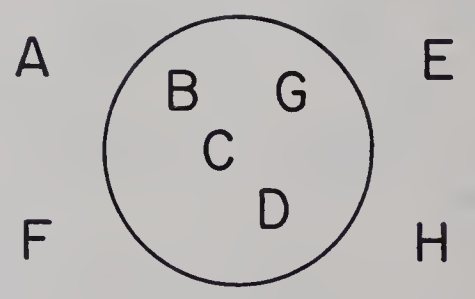
Here is another way to get a design started. Draw several rectangles on the board again. Have one child after another follow these directions. (1) Draw a line that starts at one side and goes across to the other but doesn't cross itself. (2) Draw another line that goes from one side to the other and crosses the first line. (3) Draw a loop that starts at the bottom, goes up and then comes back to the bottom (or from top back to the top). Examples might look like this:



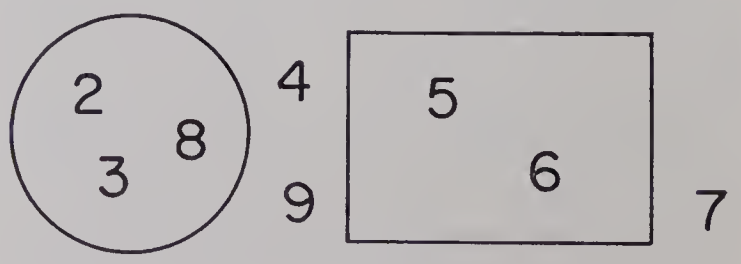
With different colored chalk, shade in any desired number of areas, regions or pastures. Now, as before, all the children can work independently. On one side of their papers, they might be asked to color three pastures or regions one color and three another color. On the other side, they might be asked to color six regions different colors, or all in the same color.

You may want to have the children write a number in the lower right-hand corner of their papers that tells how many spaces are colored—a title for the design. Drawing straight lines, curved lines, lines that cross, seeing intersecting lines create interesting regions are all ideas that will extend into the study of spaces that we call geometry. Following simple directions that lead to surprisingly different results each time, is a valuable experience.

a. Outside and inside. Explain to the class that a single closed curve of any kind establishes two *regions:* the *outside* and the *inside*. To demonstrate this, draw the following diagram on the board:

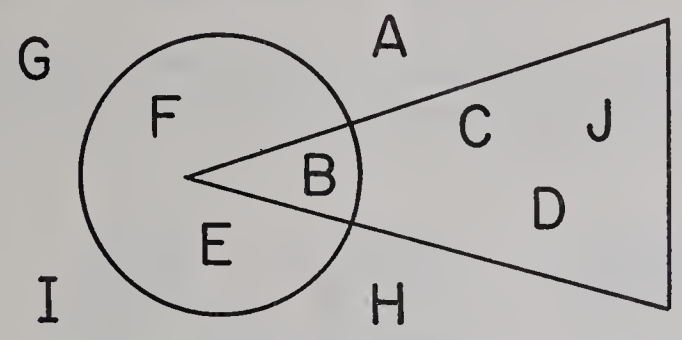


Ask, "Which letters are in the *inside region*?" Which letters are in the *outside region*?" Next, explain that the number of regions to be considered increases if more than one figure is involved. Draw the following diagram of a circle and a square on the board:



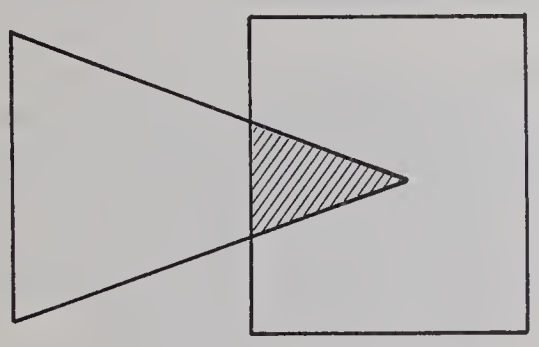
Ask, "How many numerals are in the region outside the circle? (5 and 6 are *inside the square, but they are also outside the circle.* The answer, therefore, is 6.) How many numerals are inside the square? (2) How many numerals are either inside the square or inside the circle? (5)"

Overlapping shapes make it especially necessary to be precise about what region is being discussed. Draw the diagram:



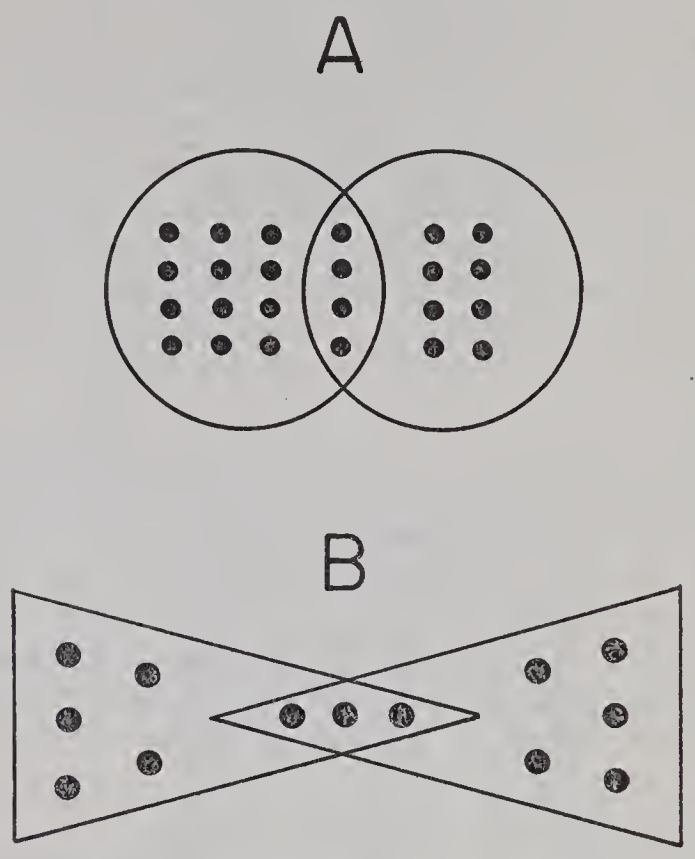
Ask, "How many letters are inside the circle? (3) How many letters are outside the circle? (7) How many letters are inside *both* the circle and the triangle? (1, "B") How many letters are *not* inside *both* the circle and the triangle? (9—all, except "B") How many letters are inside *either* the circle or the triangle? (6) How many letters are *outside* both the triangle and circle? (4) How many letters are *inside* the triangle but not inside the circle? (3)."

Draw the following diagram on the board exactly as it is shown here and have each pupil copy it:



Pupils should use a red crayon to color the region that is *both* inside the triangle and inside the rectangle. Then have them color the region that is *inside the rectangle but not inside the triangle* with a blue crayon. They should then color the region that is *outside* the rectangle with a green crayon. This latter region would include all of the triangle that does not overlap with the rectangle.

b. "I'm thinking of a region." Using a different colored chalk for each *region,* draw the diagram that is shown below on the board. Then, play the game "I'm thinking of a region," using this diagram as a model. Any region can be identified by the shape or shapes that form it and by the colored chalk used to delineate it. The object of the game is to guess the region the *leader* has in mind and to do so by asking a minimum of questions. Some pupils will discover the technique of cutting down the possibilities quickly, and will master the game easily.



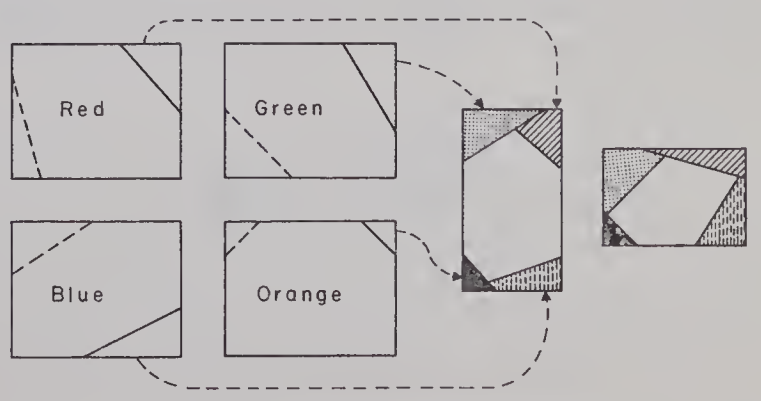
10. Making triangle designs

Suppose now for a moment that we had four pieces of colored paper—red, blue, green and orange. We could cut one corner off each sheet, some big pieces and some smaller pieces, and paste these four corner pieces on a new sheet, fitting the corners we cut off into the corners of the new sheet.

"Here is a picture of four sheets of colored paper. Who can draw a line on the sheet marked red to indicate a corner that is cut off?" Ask the same question for each sheet. Urge some children to make big pieces, small pieces, long thin pieces, etc.

"Now, let's look at the first picture frame. It's empty. What would it look like if we pasted that red corner on the picture, fitting it neatly in the corner?" Do the same with other four colors.

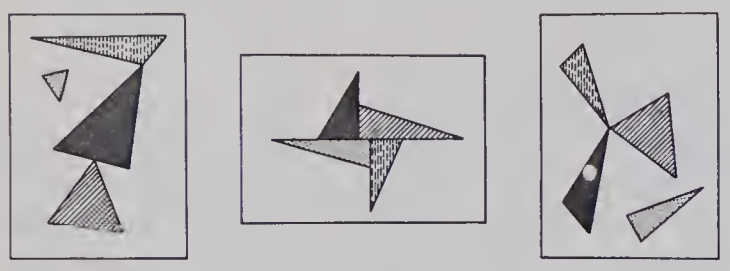
Repeat the activity, starting with a line that indicates another corner cut off the red piece. Construct a second design.



Let the children repeat the activity working either individually or in small groups.

Right triangles and similar triangles. The corners with which the children are working will later assume the dignified name of *right triangles.* The idea of *fitting* as it is used here becomes very important when we want to talk about *angles of the same size,* a condition which exists when one angle can be fitted into the other. When each of the angles of a triangle can be *fitted* into an angle of another triangle, we call them *similar.* The idea of angles fitting or not fitting is a required concept when talking about a shape or comparing shapes.

When the children have finished their corner-frames, there is a space left in each in which they can draw any pictures they like. Either at this time or later, other triangles, shapes with three sides, shapes with three corners or three points, can be cut from the remnants of the colored paper to make another kind of design. Put several examples of *triangle* designs on the board.

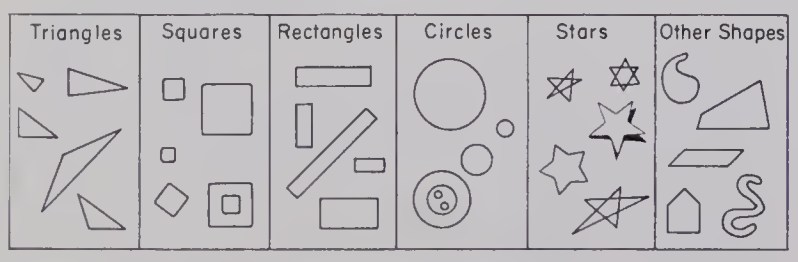


11. Drawing shapes

Section the board and write a name in each—triangles, circles, squares, rectangles, stars, other shapes.

"This section is for triangles. Who wants to draw a triangle in this section? Another triangle? A long thin triangle? A small triangle? A triangle with a very sharp point? A triangle with two very sharp, thin points?"

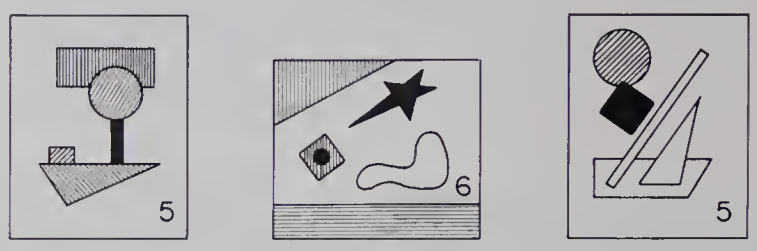
Then to *squares.* Bigger squares, smaller squares, a square inside of a square, etc. Whenever a volunteer draws a shape that doesn't belong, talk about it. "Does everyone think this shape is in the right place? Where should it go?" If it can't go in any other specific section, it can always go in the section titled "Other Shapes." No contribution need be excluded; some merely have to be redrawn in another section.



Now ask each child to draw at least one shape for every section—a triangle, square, etc. Then have them cut them out. Encourage the children to use different colored paper and to see if they can use up all of their scrap pieces.

Making designs. "Pick five of the shapes. Paste them on a sheet to make a design you like. Then take six more shapes and make another design. In the right-hand corner at the bottom, write the number that tells how many shapes there are in the picture."

On the board, sketch examples; or have some of the children sketch them. (Use several different colors of chalk.)



Have each child select a design he has made, the one he likes best. Line them up on the chalk rail or on a bulletin board. Give each a letter or number designation.

Pupils choose best design. The children are a jury. The job of each child is to study all the designs and pick the three he likes best. He is asked to write the letters or numbers of these three on a piece of paper and put it in a ballot box.

A committee is chosen to record the votes, giving one point to each picture named on each ballot. The tallying is best handled by a small group and the several designs with the highest votes are displayed prominently. (In this way, no one need be embarrassed by a public announcement that his design had none or very few votes.)

Should you discuss the three or four or five selected designs? Very briefly and not in detail, if at all. Children are good art critics but they are not able to express their basis for preferring one design to another. They respond to balance, symmetry, movement, color harmonies, etc., without having the background to verbalize such reactions.

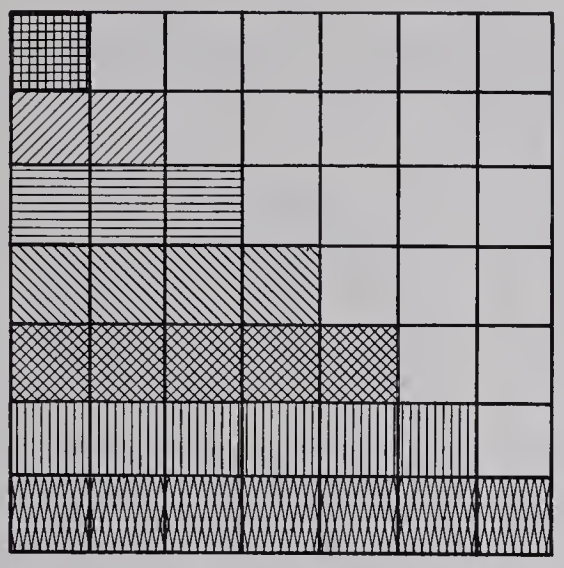
It is enough for the children to have participated in selecting favorites among a whole collection of abstract designs. There are no people or trees or cars that may or may not look like people or trees or cars to everyone. The children have made a judgment. *Likeness* was not the basis. What was the basis? You will never know, but it will be a mixture of balance, symmetry, movement, color

combinations and all the qualities that an art critic can verbalize.

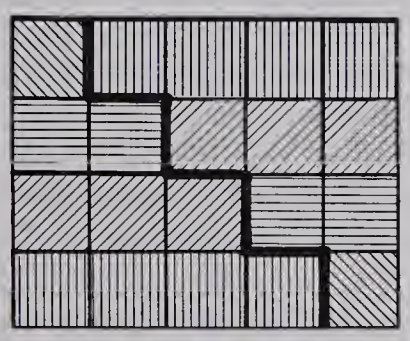
12. Graph-paper arithmetic patterns

Squared paper invites activity. It is never too early to begin; few persons will ever grow beyond graph-paper arithmetic—in mathematics, in engineering, in business, in reading the newspaper. No more effective methods have been discovered than work with graph paper to help reveal the simplest to the subtlest relationships.

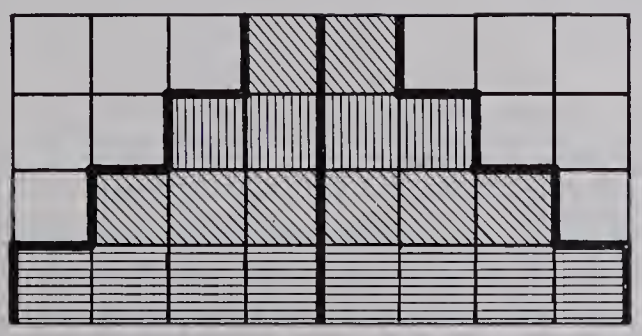
Stair-step patterns. Begin with the lowly *stair step.* Color a square with any crayon. Then the one under it and another one a different color, etc.



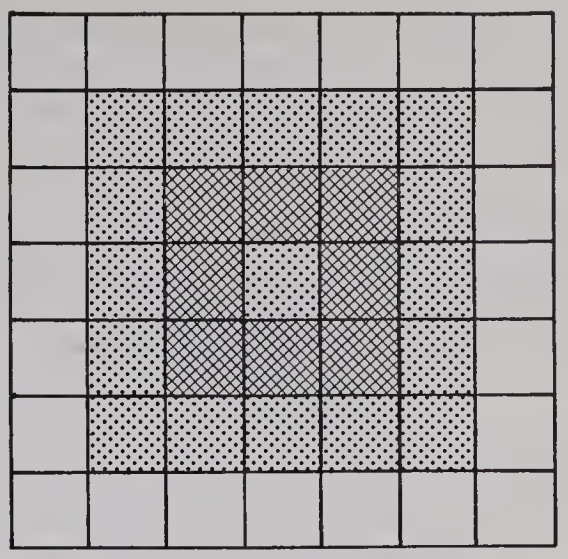
Make another set of *stair steps* up to ten high. Then, repeat the color combinations in an *upside down* set that fits on top.



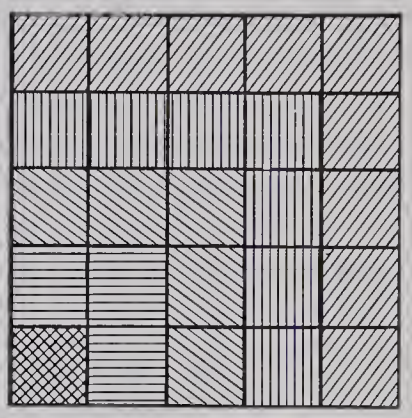
And, *back-to-back stairs.*



Nest of squares patterns. A *nest of squares* is an interesting pattern. Shade one square one color. With another crayon, color the squares around it.



Let a *nest of squares* grow out of the corner.

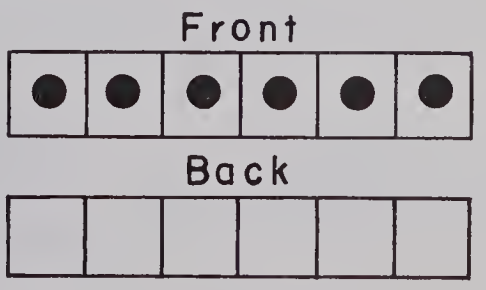


Later, this last pattern will help reveal the connection between *odd numbers* and *square numbers.* Each layer is composed of the next odd number of blocks—1, 3, 5, 7, 9, etc. Each successive square is composed of the next square number—1, 4, 9, 16, 25, etc. The pattern suggests the following table:

$1 = 1$
$4 = 1 + 3$
$9 = 1 + 3 + 5$
$16 = 1 + 3 + 5 + 7$
$25 = 1 + 3 + 5 + 7 + 9$

13. Counting activities

A most helpful teaching aid can be made easily out of paper, cardboard, or oilcloth. Establish a convenient unit, about three inches, more or less, and cut strips 2, 3, 4, 5, 6, 7, 8, 9 and 10 units long. Divide each into unit square boxes and put a large solid black circle in each unit square on one side. On the other side, mark off the unit squares, but leave them empty. The strip six units long will look like this:



Fold along each line that separates each strip into boxes so it can be easily folded along any divider. Show a strip to the class. "How many dots do you see?" Show them the back.

Now fold over 1, 2 or 3 dots. "How many do you see now? How many will you see if I turn it around?

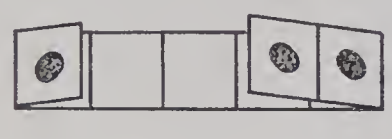
Turn it around so it looks like this:

"How many will you see if we straighten out the strip?" (Six.)

Take another strip; fold it before the children see it, and show them the side we will call the back.

"How many dots are on the side you can't see?"

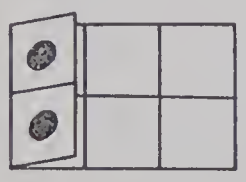
There will be five. "How many are there all together?" There will be eight. Straighten it out and count them. Then fold it differently:



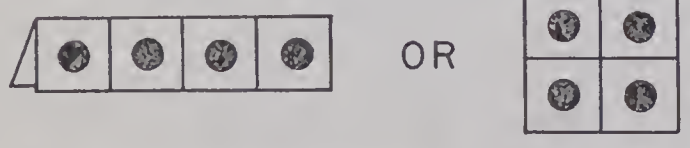
How many do you see? How many are on the other side? How many will you see if we straighten it out so you can see all the dots?"

Let different children fold the strips and display them, asking the questions. You may want to develop games with competing teams.

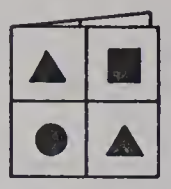
Make another set of strips, two dots deep. An example of such a piece folded might look like:



"How many dots do you see? How many are there on the other side? How many if you can see them all?" Or, it might have been folded thus:



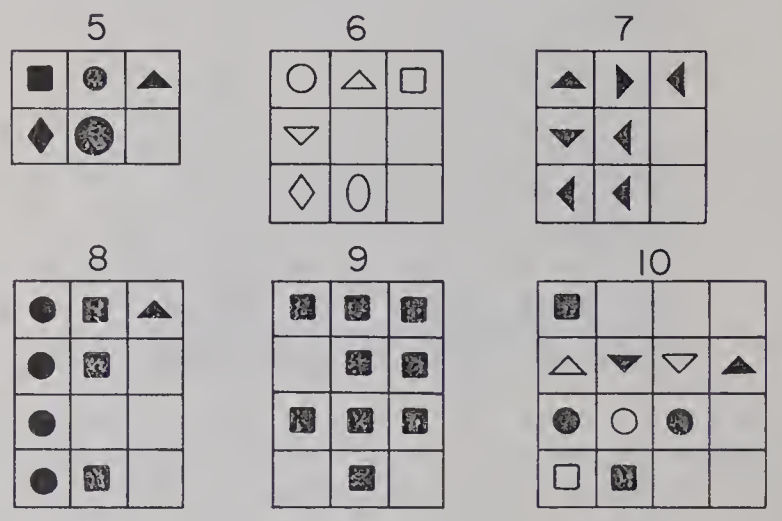
You may prefer using a variety of shapes in place of the dots:



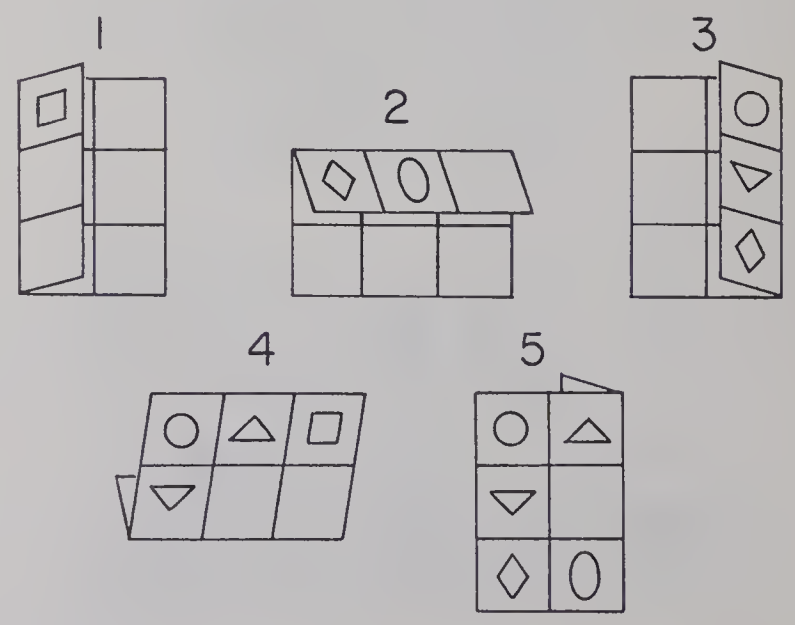
These two sets of strips and rectangular arrays involve an element of mystery, at first. "What's on the side you can't see? How many will there be when you can see them all?" As soon as the element of mystery wears off, many of the facts of arithmetic are ready in the children's minds for immediate recall.

14. Another aid for counting games

Some examples of another useful set of teaching aids are given below.



Each one of these is carefully designed so that it can be folded to display any number of the shapes desired. For example, the piece with six shapes can be folded as follows:



And, without folding, six shapes can be seen. Since each display can be folded to suggest every number from one to the total number of shapes, these few

pieces can be used to suggest every facet of addition and subtraction in which the addend is five through ten. The idea can be extended in both directions.

Show the unfolded card. "How many shapes do you see? Count them. Does everyone agree that there are nine?" Fold the card. "Now, how many do you see? Does everyone agree that we can see only four? How many will you see if I turn the folded card around? How many did you all see before I folded the card? Nine. And you see four now. How many shapes did I hide from you?"

When general agreement has been reached, turn the piece over to check the results. Appoint record keepers to keep track of each example.

NUMBER OF SHAPES

WHOLE SHEET	THE SIDE WE SEE AFTER FOLDING	THE SIDE WE CAN'T SEE
9	4	5
9	3	6
9	1	8
9	7	2

This set of aids is quite versatile. Games can be built around the cards. Small groups of children can use them, individuals taking turns as folder, record keeper and question answerer. Still another dimension can be provided in the activity. "How many empty squares are there? How many with shapes? How many altogether?" Fold the card. "How many empty squares do you see? How many did I fold over?"

And the record grid gets bigger:

BEFORE FOLDING		AFTER FOLDING					
TOTAL SQUARES	TOTAL SHAPES	SHAPES WE SEE	SHAPES WE DON'T SEE	EMPTY SQUARES WE SEE	EMPTY SQUARES WE DON'T SEE	SQUARES WE SEE	SQUARES WE DON'T SEE
12	9	4	5	2	1	6	3
12	9	3	6	1	2	4	5
12	9	1	8	2	1	3	9
12	9	7	2	1	2	8	4

15. Counting and measurement

Pupils must learn the fundamental difference between *counting and measurement.* There are two variables which make the results of any measurement an *approximation:* (1) selection of the unit of measure; and (2) observation in comparing the unit with the object to be measured. All time spent on making these facts clear will be amply rewarded by a more thorough understanding of the problems involved in using the exact *language of counting* as the *language of measurement.*

In such questions as, "How many hands wide is your desk?" whose hand is meant? Is the thumb to be included? Is the hand to be squeezed together? What if the measurement doesn't come out even? Whose opinion is to be accepted?

Or in such a question as, "How many glasses in a quart?" what size glass is to be used? How full shall the glass be? What about spilling?

The purpose of having children participate in measurement activities is to establish for them the idea of: (1) selecting a unit, and (2) observing and recording an approximate comparison.

Explain to the class that in order for them to find out how many "shoe lengths" wide the room is, (stepped off toe-to-heel-to-toe) they first will have to decide whose shoes should be used to do the measuring. Children can match shoe lengths, and select whose shoes will be used in the measuring. Pupils with "approximately" the same length shoe can check the results.

Measurement is also used in "comparing" things that cannot be placed side-by-side. For example, "Which dimension of the room is biggest?" "Is the desk higher than it is wide?" "Are two first-grade rooms about the same size?" "Are two doors about the same height?" "Who lives closest to school?" "Are all city blocks the same length?"

16. Numbers have many names

On the board, draw the following chart:

"Numbers have many names!"				
other names for 3+2	other names for 10	other names for 11-8	other names for 7+5	other names for 9-9

If you draw an odd number of columns for *other names* on the board, select an even number of teams, preferably two. A member of the first team must write on the board *another name for 3 + 2* under the proper column. A member of the next team must write *another name for 10,* and so on.

The teams should alternate each time across. The team that had 3 + 2, 11 − 8, and 9 + 9 the first time around will have 10 and 7 + 5 the next time. Each team scores whenever one of its members give a

correct other name. Whenever one of its members gives an *incorrect other name,* the opposing team scores, provided one of its members spots the error.

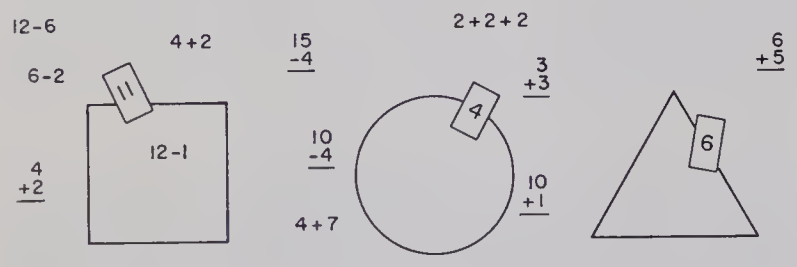
Games such as this let each child participate successfully at his own level. But most important, they drive home the fundamental concept that *numbers have many names.* This is the idea behind all *regrouping.* We can treat 32 as 20 + 12, so that 32 − 8 might be regrouped to read 20 + 12 − 8, and 20 + 4, and finally, 24.

Activities based on the idea that *numbers have many names* help the teacher to estimate the ability of each pupil. They help the teacher to discover who in the class is slow or quick, timid or confident. For example, Miss or Mr. *Way Out Ahead* may reveal themselves by using 3000 ÷ 1000 or $\sqrt{9}$ as other names for 11 − 8. (It is always possible to give the more apt pupils an opportunity to show off their prowess by asking them to write on the board the most difficult looking names for a given number that they can think of.)

A variation can be used to make the board game more difficult. A member of one team starts a name for the number in question, and the other team must complete it. Another name for 3 − 2 might be started as "0 + _" or "19 − _." The next player has to complete the name: "0 + 1," or "0 + 10 − 9," or "0 + 1 × 3 − 2," etc.

Sometimes, *you* may want to begin the other names in order to match the challenge of the situation to the recognized ability of the next player. This is important to the pupil: to be asked a question that is not too easy nor too hard, a question he can answer if he makes an effort.

A second activity. Draw this on the board:



As each expression is discussed, cross it out and rewrite it in the appropriate shape. Be as certain as possible that everyone understands what he is expected to do. Only if the child has no questions about the activity will you be able to judge successfully whether or not he is familiar with the arithmetic involved.

A third activity. Next, have class members tackle the following problem: "How many names for each of the numbers from 0 through 10, can you write in the form of ____ + ____ = ____?"

0	1	2
0 + 0 = 0	0 + 1 = 1	2 + 0
1 name	1 + 0 = 1	1 + 1
	2 names	0 + 2
		3 names

3	4	5

Then, have them record the results in a chart.

Number	Names in form of ____ + ____ = ____	
0	1	
1	2	
2	3	
3	4	
4	5	
5	6	
6	7	
7	8	
8	9	
9	10	
10	11	

This leads to a nice pattern. It suggests that there will be 21 names for 20 in the form of ____ + ____ = ____. Test the pattern. Does it keep on giving you reliable results?

A fourth activity. Next, you should explore the same question for names that are in the form ____ + ____ + ____ = ____.

0	1	2
0+0+0 = 0	0+0+1 = 1	0+0+2 = 2
(1 name)	0+1+0 = 1	0+2+0 = 2
	1+0+0 = 1	2+0+0 = 2
	(3 names)	1+1+0 = 2
		1+0+1 = 2
		0+1+1 = 2
		(6 names)

3	4	5
0+0+3 = 3	0+0+4 = 4	0+0+5 = 5
0+3+0 = 3	0+4+0	0+5+0
3+0+0 = 3	4+0+0	5+0+0
0+1+2 = 3	0+1+3	0+1+4
1+0+2 = 3	0+3+1	1+0+4
0+2+1 = 3	1+0+3	0+4+1
1+2+0 = 3	1+3+0	1+4+0
2+1+0 = 3	3+0+1	4+1+0
2+0+1 = 3	3+1+0	4+0+1
1+1+1 = 3	2+2+0	0+2+3
(10 names)	2+0+2	2+0+3
	0+2+2	0+3+2
	1+1+2	2+3+0
	1+2+1	3+0+2
	2+1+1	3+2+0
	(15 names)	3+1+1
		1+3+1
		1+1+3
		2+2+1
		2+1+2
		1+2+2
		(21 names)

The results, once more, are to be recorded in a chart.

Numbers	Names in form of ___+___+___=___	
0	1	
1	3	
2	6	
3	10	
4	15	
5	21	
6	?	

A fifth activity. Each of the following problems presents a pattern of its own:

1) "How many names for each of the numbers 1 through 10 can you find, using any one pair in one order only and omitting 0?"

0 → none

1 → none

2 → 1+1 (1 name)

3 → 2+1
1+1+1
(2 names)
(1+2 does not count, because 2+1 has already been included)

4 → 3+1
2+2
2+1+1
1+1+1+1
(4 names)

5 → 4+1
3+2
3+1+1
2+2+1
2+1+1+1
1+1+1+1+1
(6 names)

6 → 5+1
4+2
4+1+1
3+3
3+2+1
3+1+1+1
2+2+2
2+2+1+1
2+1+1+1+1
1+1+1+1+1+1
(10 names)

7 → 6+1
5+2
5+1+1
4+3
4+2+1
4+1+1+1
3+3+1
3+2+2
3+2+1+1
3+1+1+1+1
2+2+2+1
2+2+1+1+1
2+1+1+1+1+1
1+1+1+1+1+1+1
(14 names)

8 → (21 names)

9 → (29 names)

10 → (39 names)

Pupils will probably discover for themselves that it is helpful to devise a system for writing down the different names of each number. Otherwise it becomes difficult to keep track of the combinations. (Note: do not be disturbed if you cannot predict the number of combinations possible for 11. It took two famous mathematicians a long time to develop a formula that allows such a prediction to be made, and the formula is a complicated one.)

2) "Find all the names for the *even* numbers 1 through 10, using the sum of *even* numbers only and omitting 0:

4 → 2+2 (1 name)

6 → 4+2
2+2+2 (2 names)

8 → 6+2
4+4
4+2+2
2+2+2+2
(4 names)

10 → 8+2
6+4
6+2+2
4+4+2
4+2+2+2
2+2+2+2+2
(6 names)

3) Find the names for all the numbers 2 through 10, using the sum of *odd* numbers only, and omitting 0:

2 → 1+1 (1 name)

8 → 7+1
5+3

(more, next page)

3 → 1+1+1 (1 name)
4 → 3+1
1+1+1+1 (2 names)

5 → 3+1+1
1+1+1+1+1(2 names)

6 → 5+1
3+3 (4 names)
3+1+1+1
1+1+1+1+1+1

7 → 5+1+1
3+3+1 (4 names)
3+1+1+1+1
1+1+1+1+1+1+1

9 → 7+1+1
5+3+1
5+1+1+1+1
3+3+3 (7 names)
3+3+1+1+1
3+1+1+1+1+1+1
1+1+1+1+1+1+1+1+1

	Names
2	1
3	1
4	2
5	2
6	4
7	4
8	6
9	7
10	10

17. Names for numbers in different forms. How many names could be written for zero in the next green box above 1, there are two possibilities. In completing this row of green boxes, a familiar pattern becomes apparent: one name for 0; two names for 1; three names for 2; four names for 3; five names for 4.

As observed earlier, a very different condition develops when the form becomes: "__ − __." Now there are unlimited possibilities.

Below the number line, the form "__ × __" appears. The pattern here is quite different: There are an unlimited number of names for 0 in the form "__ × __"; one name for 1 in the form "__ × __"; two names for 2 in the form "__ × __"; two names for 3 in the form "__ × __"; three names for 4 in the form "__ × __."

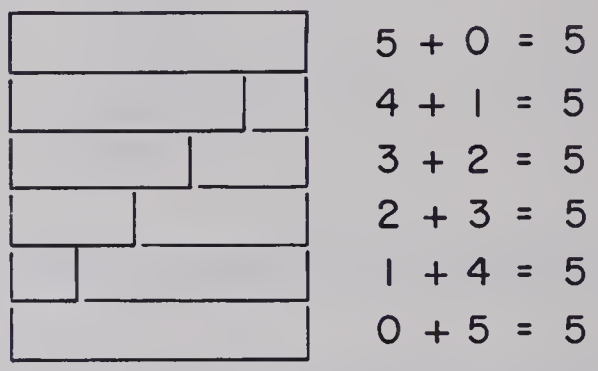
In this table, when there are only two names for a number in the form of "__ × __," the number is a prime number (a number that is divisable by exactly 2 different numbers, itself and 1): 2, 3, 5, 7, 11, 13, etc.

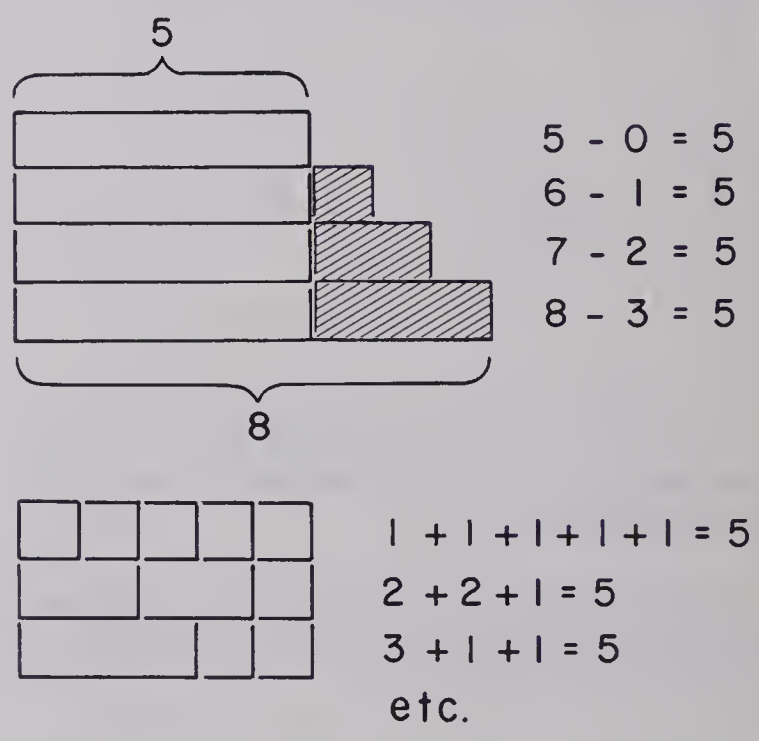
Finally, we discover that the form of "__ ÷ __" yields an unlimited group of names for each number. While there are three ways given here of writing these names, our main interest is in the form which leads so naturally into fractions: $\frac{1}{1}$, $\frac{6}{3}$, $\frac{3}{1}$, etc.

Care should be taken with other names for 0 in the form of division. It has to be asserted that division by 0 doesn't make any sense in arithmetic; it must be excluded. The only permissible names are 0 divided by any number other than 0: $\frac{0}{2}$, $\frac{0}{1}$, $\frac{0}{3}$, $\frac{0}{4}$, etc.

Manipulative materials. Give each student a series of paper or cardboard strips which he can manipulate. (A 1-inch square for one, a 1 × 2 strip for two, a 1 × 3 strip for three, etc.) Blocks or rods in a series of lengths representing 1 to 10, which you may already have in your classroom, will serve the same purpose. These can be used in a variety of ways, not only to reconstruct the sizes and relationships on these pages, but also to work out such problems as, "How many names has a number?"

For example, pupils can make "little pictures" of 5 in such ways as:





Fractions. "Is two dollars divided by two the same as one dollar?" ($\frac{2}{2} \rightarrow 1$) "If we divide 4 sandwiches by 2, do we have two sandwiches? Are 6 sandwiches divided by 3, two sandwiches?" ($\frac{4}{2} \rightarrow \frac{6}{3}$) "Would one sandwich divided by two be the same as 3 sandwiches divided by 6, or 2 sandwiches divided by 4?" ($\frac{1}{2} \rightarrow \frac{3}{6} \rightarrow \frac{2}{4}$)

If the dime is used as a basic unit, a nickel is to be thought of as $\frac{1}{2}$ a dime, a quarter as $2\frac{1}{2}$ dimes, a half dollar as 5 dimes, fifteen cents as $1\frac{1}{2}$ dimes. Continuing with the dime as the basic unit, three nickels would be called $3\frac{1}{2}$ dimes, and a quarter and a nickel would be $2\frac{1}{2}$ dimes and another $\frac{1}{2}$ dime."

In terms of dimes, the following could be accurately interpreted: $\frac{2}{2} = 1$; $\frac{1}{2} + 2\frac{1}{2} = 3$; $\frac{3}{2} = 1\frac{1}{2}$; $2\frac{1}{2} = \frac{5}{2}$.

Each time pupils answer questions like these, the mathematical symbol should be written on the board. The arrow, to be read "suggests" or "is the same idea as," is preferred in situations like this, to "=" ("equals"). The idea has not yet been firmly established that fractions are actually numbers or can be treated as numbers. Pupils do not know that they can add, subtract, multiply, and divide fractions. What pupils have been learning is that $\frac{4}{2} = \frac{6}{3}$; and that these are two names for the same number.

18. Row and column activities

The game, "Row-Column," can be adapted to any level of maturity. Use a "magic square," for example.

	A	B	C
X	8	1	6
Y	3	5	7
Z	4	9	2

A player for "Row" chooses X, Y or Z, and writes it down secretly. Then a "Column" player selects A, B or C. The players announce their choices. The number written where the announced row and column cross becomes "Row's" score.

Next a player for "Column" starts by choosing A, B, or C and secretly writing it down. Then a "Row" player selects X, Y, or Z. "Column's" score is discovered by finding the number where the choices of the two players cross.

Each person or team keeps a cumulative score. The first to reach a predetermined total, such as 25 or 50, wins.

Other grids can be made by adding, subtracting or multiplying the number in each cell of the magic square by one constant number:

Add 2

10	3	8
5	7	9
6	11	4

Subtract 1

7	0	5
2	4	6
3	8	1

Multiply by 2

16	2	12
6	10	14
8	18	4

In the examples above, the numbers in the rows and columns have been selected so that the sum for any three numbers down or across is the same. Neither "Row" nor "Column" is favored; chance for a high score is even. You may use smaller numbers on the square:

	A	B	C
X	1	2	1
Y	0	1	3
Z	3	1	0

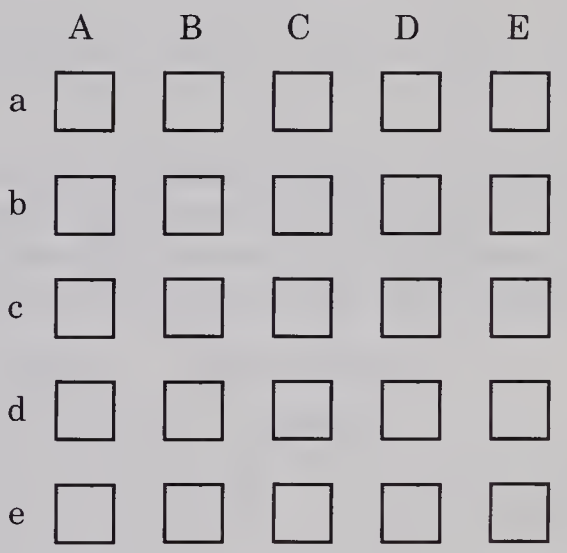
	D	E	F
U	0	1	2
V	3	0	0
W	0	2	1

	G	H	I
R	1	2	3
S	5	0	1
T	0	4	2

19. Exploring the magic square

More capable children may want to explore the principle of the magic square. In the first magic square shown above, the rows, columns and principal diagonals each total 15. If 6 were added to each cell, the new magic number would be 33; if each cell were multiplied by 3, the magic number would be 45.

Using notations. Put a diagram such as the following, on the board:



The notations B (AB,a) would refer to the number of boxes in colums *A* and *B* that are also in row *a* or, 2. A diagram such as the one above might represent seats in a classroom.

As soon as a pupil understands the idea clearly, he can look at a notation such as, N (ABD,ce) and know that the number N stands for is 6, because an array of 3 columns and 2 rows includes 6 items.

N(BC,ab) + N(D, abc) = 7 because 4 + 3 = 7. Later on, this can be interpreted as $(2 \times 2) + (1 \times 3) = 7$. Please do not suggest to pupils that this is an example of algebra. It is nothing of the kind. It is simply a kind of shorthand used in talking about a picture, such as the one on the board. Algebra is also a shorthand, but it is a different kind of shorthand.

Whenever you can, use notations in talking about a picture, an example, or a problem. Just remember that any notation is of use only when an agreement has been made about the meaning of the notation. Such notations (or shorthand signs) as "6," "10," "+," "×," "=," etc. are very useful because there is general agreement about what they mean in an arithmetical context.

More grids. Using any letters you want, create a grid similar to the following:

	a	b	c	d	e	f	g
h	B	B	B	E	E	E	E
i	B	B	B	E	E	E	E
j	F	F	F	E	E	E	E
k	F	F	F	P	P	P	P
l	F	F	F	P	P	P	P
m	F	F	F	P	P	P	P
n	F	F	F	P	P	P	P

According to the above grid, "N (ab,ijk)" means that the number of letters in columns *a* and *b,* and in rows *i, j* and *k* is 6. (There are 2 B's and 4F's.) A chart, based on the above grid, could be drawn:

	Total number of letters	Number			
		B's	E's	F's	P's
N (ab,ijk)→	6	2	0	4	0
N (cde,h)→	3	1	2	0	0
N (fg,mn)→	4	0	0	0	4
N (bcd,lmn)→	9	0	0	6	3

To vary the exercise, fill in the five columns with appropriate figures and ask pupils to give a notation that could be written in the left-hand column. For example:

?	8	2	4	2	0
?	4	1	1	1	1

Additional activities can be given to the children that finish the page first. This frees the teacher for individual work. Another grid that might be used is of shapes:

	a	b	c	d	e	f
g	□	□	□	△	△	△
h	□	□	□	△	△	△
i	•	•	•	△	△	△
j	○	○	○	○	○	○
k	○	○	○	○	○	○

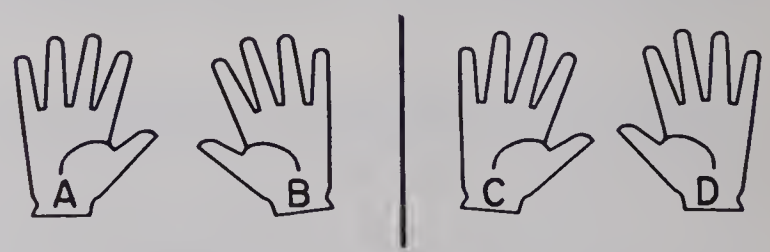
20. Multiplication activities

The series of activities that follows could be repeated in many different ways. Select the examples you like best, and repeat them often. Through these experiences, pupils will gain a real understanding of "multiplication."

a. Pairing digits. Put the chart below on the board. Fill in the "Digit on Left" and "Digit on Right" columns, as you go. Pupils should give you the answers to put in the other columns.

Digit on Left	Digit on Right		Total
2	0,3	20, 23	2
1, 2	1, 2	11, 12, 22, 21	4
1, 2, 3	5	15, 25, 35	3
7, 8, 9	1, 2	71, 72, 81, 82, 91, 92	6
1, 2, 3,	4, 5, 6	14, 15, 16, 24, 25, 26,	9
		34, 35, 36	

b. Pairing gloves. State the problem in this way: "Mary's mother bought her two pairs of gloves that were alike. In how many ways can she put them on?"



AC, AD, BC, BD

"If Mary lost one, in how many ways can she make a pair of gloves? Suppose she lost B?" (AC and CD.)

"Suppose she lost C, as well as B?" (AD only.)

"Suppose she found B and lost A as well as C?" (None—for she would have two gloves for the same hand.)

c. Couples. Here is another problem statement: "There were three men and three women sitting together. Jim said his parents were in the group. He wanted John to guess which couple his parents were. One way for John to find out would be to try all the possible combinations. But to do this he would have to give the men and women letter names in order to make a chart of the possibilities. The chart would look like this:

Men	*Women*
Mr. *A*, Mr. *B*, Mr. *C*	Mrs. *X*, Mrs. *Y*, Mrs. *Z*

"The possible combinations would be: *AX, AY, AZ, BX, BY, BZ, CX, CY, CZ*

"If John knew that Mr. *A* was not Jim's father, how many possibilities are there? (6-*BX*, *BY*, *BZ*, *CX*, *CY*, *CZ*.)

"If John knew that Mrs. *X* was not Jim's mother, how many possibilities are there? (4-*BY, BZ, CY, CZ*.)

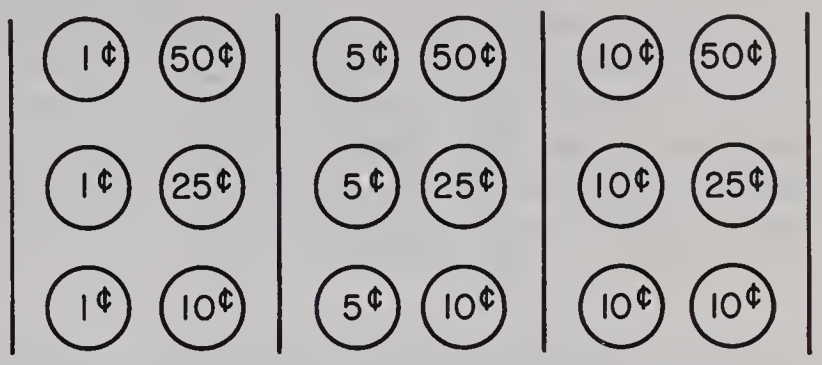
"If Jim said, 'Mr. *C* is not my father,' who would his father be?" (Mr. *B*.)

"If Mrs. *Y* is not Jim's mother, who is his mother?" (Mrs. *Z*.) "Who are Jim's parents?" (Mr. *B* and Mrs. *Z*.)

d. Pairing coins. You need two boxes and some coins. In one box put one penny, one nickel, and one dime. In another, put one dime, one quarter, and one half dollar. Have individuals from the class close their eyes and pick coins from the boxes. Make a record of the coins and total amount of money each child has picked:

Name	Box A	Box B	Total
Mary	1c	10c	11c
John	5c	25c	30c
Bill	10c	50c	60c
Barbara	5c	10c	15c
Alex	10c	10c	20c
Jean	5c	50c	55c
Bob	1c	50c	51c
William	10c	25c	35c
Helen	1c	25c	26c

The above chart includes the nine combinations that are possible. After you have recorded the first nine draws, ask if there are any repetitions. If there are, eliminate them and continue as before. When the class has come up with six or seven unique combinations, ask, "Can you think of any other possible combinations that we haven't had yet?" Pupils may find it easier to find combinations if they make a systematic record of the possibilities:



Repeat this same activity with fewer than three coins. Keep reducing or changing the number of coins—2 and 3, 3 and 2, 2 and 2, 1 and 2, 2 and 1, 1 and 1, 0 and 3, 2 and 0. When a record (table) is finished, erase the amounts in column A and B. Ask, "Can we tell which coins each child drew if we know only the total amount?" (Of course!)

e. How many combinations? This time, you must—Take three boxes. Place a penny and nickel in A; a nickel and dime in B; a dime, quarter, and half dollar in C. Have pupils draw a coin from each box. There are 12 possible combinations (because $2 \times 2 \times 3 = 12$) and the total amount represented by each draw will be different or unique. The following chart will result (do not include the right-hand column until the idea of multiplication has emerged):

A	B	C	Combinations	Because
2	2	3	12	$2 \times 2 \times 3 = 12$
2	2	2	8	$2 \times 2 \times 2 = 8$
2	1	3	6	$2 \times 1 \times 3 = 6$
2	1	1	2	$2 \times 1 \times 1 = 2$
1	2	1	2	$1 \times 2 \times 1 = 2$
1	1	3	3	$1 \times 1 \times 3 = 3$

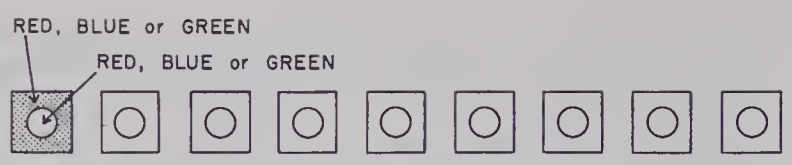
If you use coins in the pattern suggested, every amount will be unique, so the selections can be reconstructed from a column showing only the amounts.

f. How many ways to color? The question here is: How many ways are there to color the design below, if the following conditions are met:

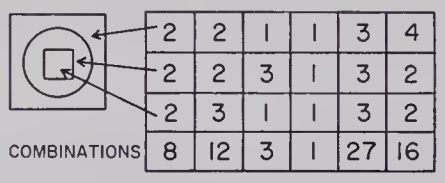
A—The area between the square and circle is to be red, blue or green.

B—The area inside the circle is to be red, blue or green.

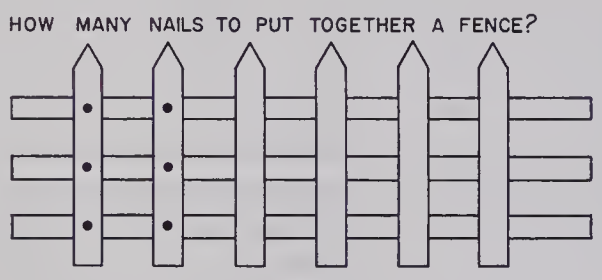
C—Both areas must be colored.



There are nine ways (because $3 \times 3 = 9$). Repeat the activity using three colors and two colors (result —6 combinations). To make the activity more difficult, increase the number of colors: three colors and four colors, for example, which produce 12 possible combinations. This activity can be extended to a design having three areas:



g. How many nails? Here is a seventh activity: Ask the class to tell you how many nails it is going to take to put together a fence like this:



Cross Pieces	Uprights	Nails
3	2	6
3	3	9
3	4	12
3	5	15
3	6	18

h. How many combinations of money? And an eighth—This example is planned to help develop the idea of multiplication and, at the same time, to provide pupils with an exercise in writing amounts of money. Ask pupils how many different amounts of money could be made by combining one each of the following paper money and coins:

Bills for $1.00, $5.00, and $10.00 and coins for 1c, 5c, 10c, 25c, 50c (Answer: 15, because 3×5=15)

$1.01	$1.05	$1.10	$1.25	$1.50
$5.01	$5.05	$5.10	$5.25	$5.50
$10.01	$10.05	$10.10	$10.25	$10.50

Bills for $1.00, $5.00, and $10.00 and coins for 5c, 10c, 50c (Answer: 9, because $3 \times 3 = 9$)

Try examples in which there is either no bill or no coin. In such cases there can be no combinations. This should serve to suggest the idea that $0 \times 3 = 0$, $2 \times 0 = 0$, etc.

i. How many batteries? From the world of sports—A baseball team has four pitchers and four catchers. How many different pitcher-catcher combinations can they plan on using? (A pitcher-catcher combination is called a "battery." Another team has four pitchers, and three catchers. How many combinations *less* than the first team can they make? (Four less—$4 \times 4 = 16$, $4 \times 3 = 12$ and $16 - 12 = 4$.) The chart below records the number of possible combinations that can be made if the number of pitchers is from 0 – 4 and the number of catchers if from 0 – 4.

	Pitchers				
Catchers	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	6	8
3	0	3	6	9	12
4	0	4	8	12	16

j. How many combinations of fingers? The question: "In how many ways can you combine a finger of your right hand with a finger of your left hand?"

Make the grid shown below. Ask a pupil to make a combination. Record it. Next, ask a child to make a different one. Record that. Etc.

	Right Hand				
Left Hand	Thumb	Fore	Middle	Ring	Little
Thumb					
Fore					
Middle					
Ring					
Little					

There will, of course be 25 combinations.

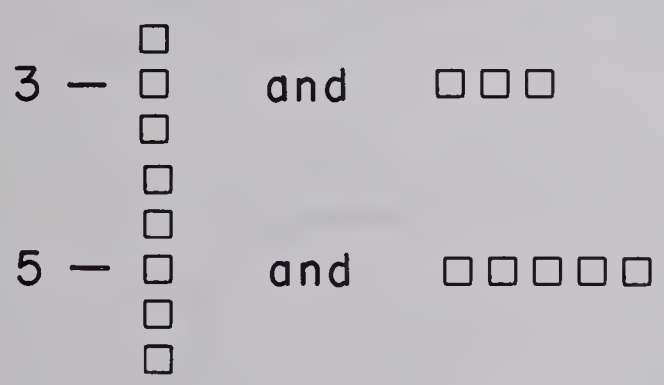
"What happens if we don't count thumbs?" (Only 16 combinations.) "Suppose we don't count thumbs or little fingers?" (Only 9 combinations.) Whenever a finger is eliminated from *each hand,* the number of possible combinations is still the square number of the fingers on any one hand: 16, 9, 4, 1, 0 (4×4, 3×3, 2×2, 1×1, 0×0).

k. Column and row arrangements. The problem is: Can you arrange 12 chairs in rows and columns so that each row has the same number of chairs and each column has the same number of chairs?

Total Chairs	In each Row	In each Column
12	3	4
12	4	3
12	2	6
12	6	2
12	1	12
12	12	1
16	4	4
16	8	2
16	16	1

If 24 chairs were used, it would be possible to illustrate: 1×24, 24×1, 12×2, 2×12, 8×3, 3×8, 6×4, 4×6. Other illustrations of multiplication can be shown by varying the number of chairs.

"In compliance with the rule that the number of chairs in each row and column be equal, what number of chairs would have to be set up so that there was only 1 row or 1 column?" (2, 3, 5, 7, 11 13, 17, 19, 23, etc.)



"These numbers are different from others. They are called prime numbers." Ask, "Can you find more *prime numbers*?"

l. How many first flights? You are a travel agent: In planning a trip, Mr. Traveler wanted to leave from New York, Boston, or Washington, D.C. He wanted to visit London, Paris, or Rome first. In how many ways could he plan his trip?

	Leaving from:		
Arrive	NewYork	Boston	Washington, D.C.
London	NY - L	B - L	W - L
Paris	NY - P	B - P	W - P
Rome	NY - R	B - R	W - R

Vary the number of cities Mr. Traveler might want to leave from, and the number of cities he might want to visit first.

m. How many combinations of food and drink? A menu:

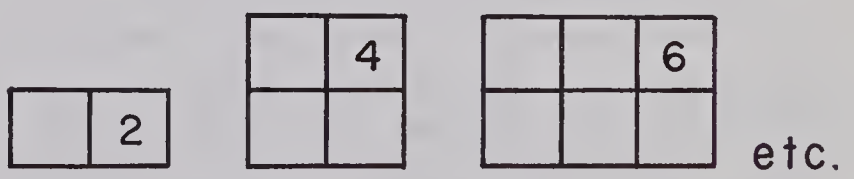
Sandwiches—	Hamburger	25c
	Tuna Fish	19c
	Hot Dog	17c
	Cheese	11c
Beverages—	Regular Milk	10c
	Chocolate Milk	13c
	Soda	6c

Jim had a sandwich and a beverage. His bill was 25c. What did he have? In how many ways could he have ordered a sandwich and a beverage from the same menu (12)?

	Reg. Milk 10c	Choc. Milk 13c	Soda 6c
Hamburger 25c	35c	38c	31c
Tuna Fish 19c	29c	32c	25c
Hot Dog 17c	27c	30c	32c
Cheese 11c	21c	24c	17c

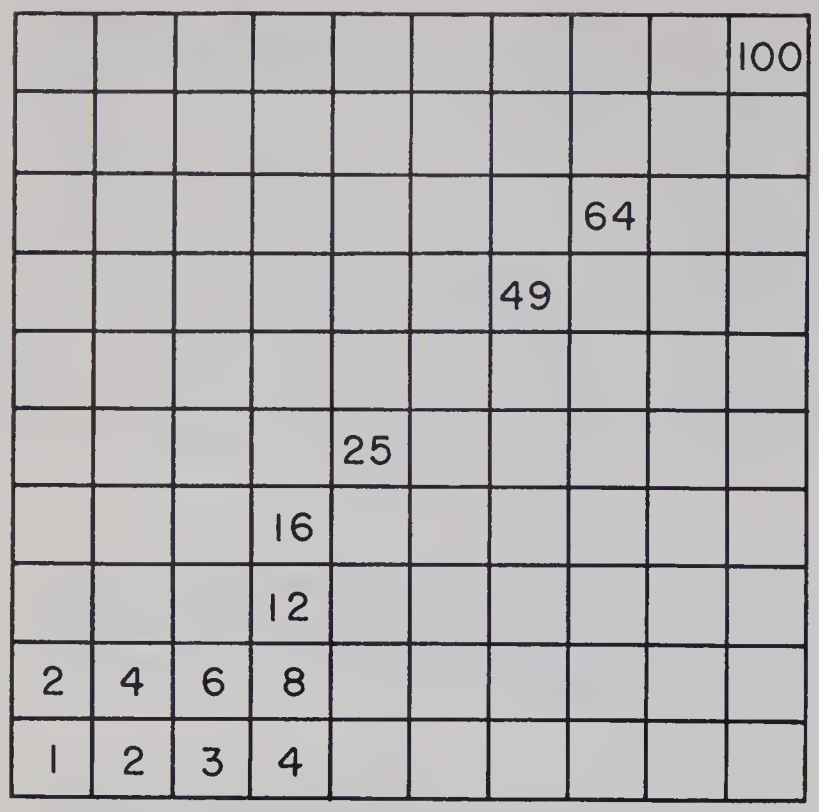
Vary the menu in numbers of items and prices. Be careful not to use prices that result in the same amount for two different combinations. One easy method is to add or multiply all the items by one figure. In this way you preserve the pattern of uniqueness worked into the example given.

n. How many squares? With scissors, you make a machine: From paper that is squared on one side, cut as many rectangles as possible, each containing a number of squares. No side should be more than the length of ten squares.



In the upper right-hand corners, record the number of squares. Arrange them in order of size. Stack them so the bottom left corners are all together—putting the one with the largest number down first, as shown at the top of the next column.

Wherever there is a square without a number, a rectangle can be cut to fill the gap. (You may want to start with all rectangles whose sides are not more than the length of five squares—so that 25 will show in the upper right-hand corner.) You have created what we call a *multiplying machine.*

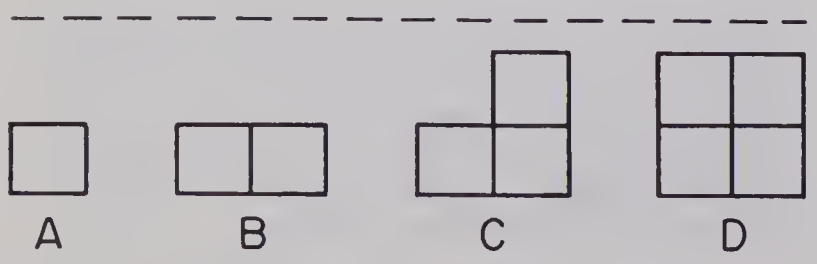


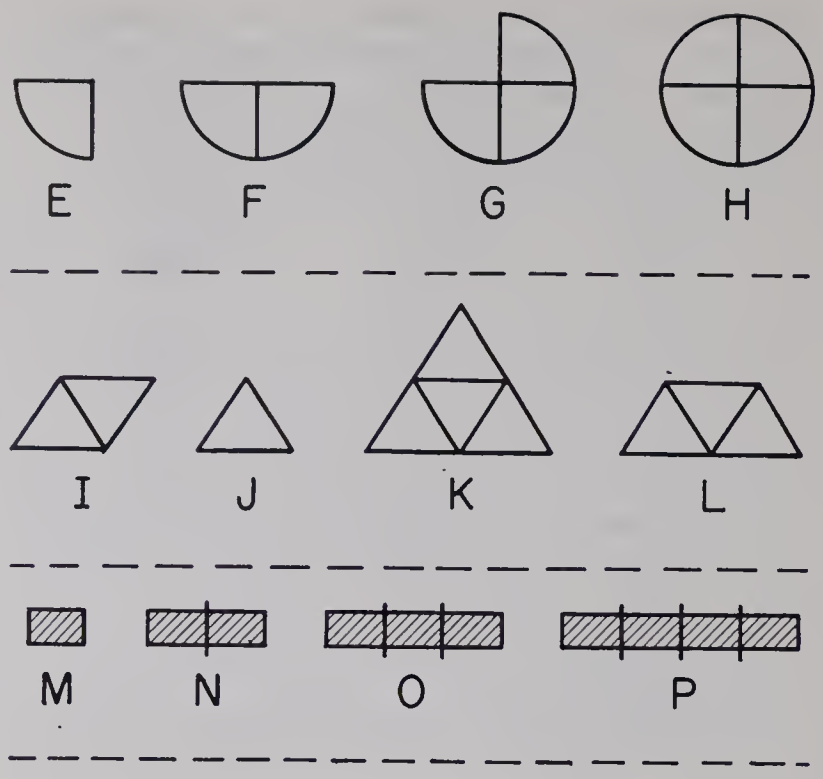
Put the rectangles face down on a table. Ask, "Who can find the rectangle divided into 4 parts? Who can find the 9 squares, 16 squares, 25 squares, etc. What shape are these parts?" (Squares; and again we come upon the square numbers – 0, 1, 4, 9, 16, 25, 36, 49, etc.) Teams can play a game. One team says, "Find 24." If the other team finds "24," it belongs to them. If not, the first team gets a chance to find it. Each team collects the rectangles it has correctly picked. Score is kept by adding the *squares* of the rectangles each team has collected.

Each team collects the rectangles it has correctly picked. Score is kept by adding the *squares* of the rectangles each team has collected.

o. Shapes as units in multiplication. We know that "*If* . . . , *then* . . ." is the most used expression in mathematics. "*If* John has ten marbles and loses seven, *then* he must have three left." "*If* I draw a line from a point inside a triangle to a point outside a triangle, then I must cross the triangle." "*If* 19 is more than a first number and that number is more than a second number, *then* 19 is more than the second number." "*If* we agree that the size of a shape such as a certain square suggests five, *then* a shape twice as large suggests 10."

Many similar activities can be developed around diagrams drawn on the board:



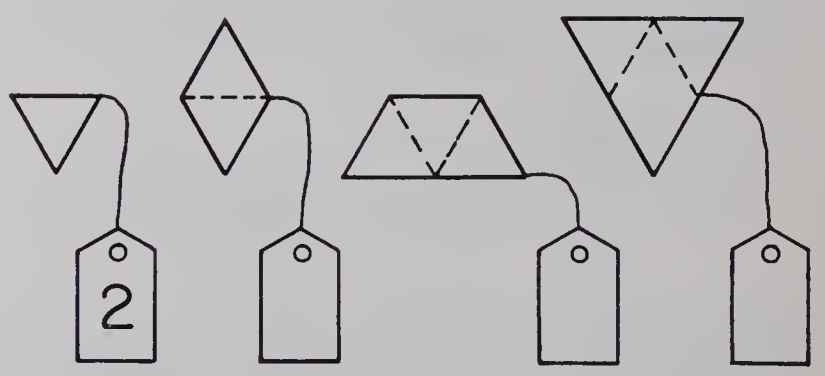


"*If* the size of A suggest five cents, *then* what is suggested by the sizes of B, C, and D?" B could suggest two nickels, one dime, or ten pennies.

"If the size of H suggests a dozen eggs, then what do the sizes of E, F, and G suggest?"

The designs used to reflect relative sizes may be any shape, any order, but not confusing.

Next, draw this diagram on the board:

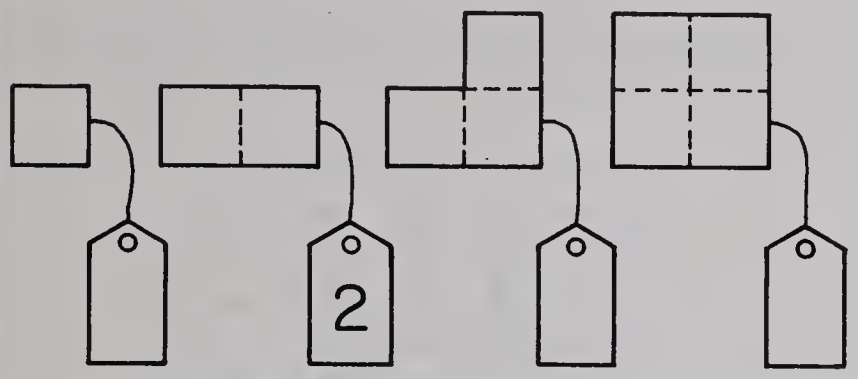


"*If* we say that the size of the first shape suggests *two* as the label indicates, *then* how would we label the other shapes?" The tags would read *2, 4, 6,* and *8.* Next, erase the *2,* the *6,* and the *8,* leaving the *4.* Ask pupils, "Suppose to start with we were given the number 4 in the second tag. How would we number the other tags?" The same system of labeling would result: *2, 4, 6,* and *8.* Repeat the activity, starting with only the tag labeled *6*; then with only the tag labeled *8.*

Next, erase all labels and put *6* in the second tag. "If the tag on the second shape were marked with a 6, *then* how would we label the tags for the other shapes? Is there a shape twice as large as the one we

have labeled with a 6?" (Yes) "What should its label be?" (12) "Now, how do we label the other tags?" (3 and 9) By this time, most of the class will have the idea and will be able to proceed with the page. Avoid any temptation to verbalize the relationship between shapes in some way such as: "The smallest tag should be half of the next one." Either a child has understood the activity, or he has not. If he has, then he can label the pictures properly. If not, he can't label them. There need be no verbalization to determine this question.

If some children are still confused, draw another diagram on the board, such as:



Repeat the development as already outlined. There is a subtlety in this activity that may properly disturb some children. They are used to counting *things*; a dot is a dot, regardless of its size. Therefore, it is necessary to make it clear that the ground rules have been changed. Emphasize the fact that we *are* concerned here with size. Just as it is possible to arrange collections of things according to the number of objects in them, it is also possible to arrange objects according to their size.

We have gone even further than a simple consideration of grouping objects according to size, by introducing the idea of *relative size.* If children have not absorbed this idea from the examples, keep drawing more pictures, more illustrations. The fact is that the child who is distrubed by the change in *rules* may be the most mathematically inclined. In any event, it is wise to accept the responsibility for introducing the new idea by taking great care in making it clear to the pupils.

Another possible approach is, "Suppose we have some pieces of paper of different sizes:

"Here are some small blocks. Now, how many blocks do you need to cover each piece of paper?" Keep a record of the results. Then put the blocks away.

"Suppose we had blocks of all the same size. If two of them would just cover the next-to-the smallest piece of paper, then how many would be required to cover the other pieces of paper?"

Don't by-pass or overlook the slowest child. The ability to make an agreement involving relative sizes is important in later problems of multiplication, area, numeration, fractions, etc.

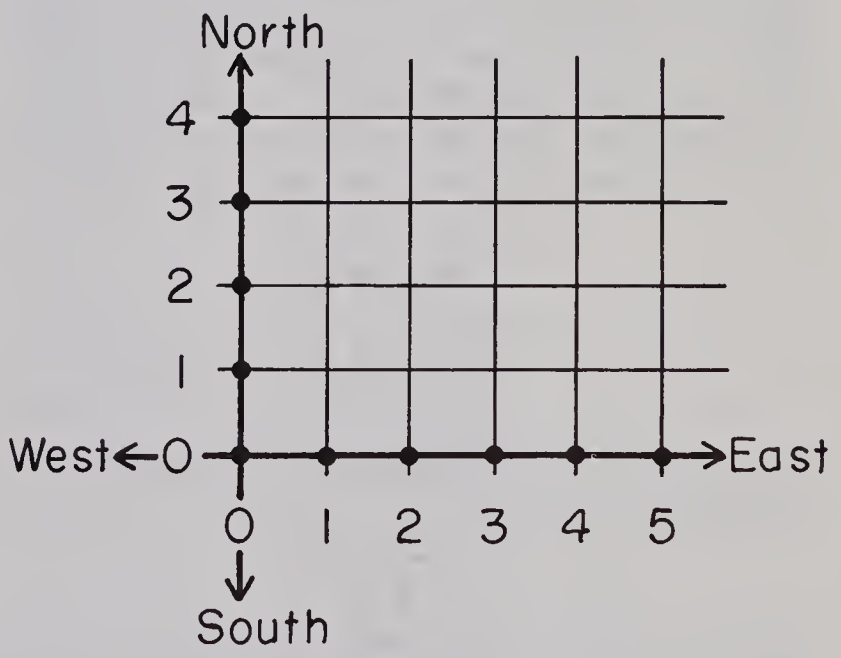
p. Using grids as a multiplication activity. Draw on the board two number lines that intersect, as shown below. Label each line beginning with "0" at the point of intersection.

Ask pupils, "How many jumps is it from 2 on one line to 3 on the other? How many jumps is it from 4 on one line to 4 on the other?" Record the results.

Starting point on one line	2	4	3	2	1	4	3
Landing point on the other	3	4	2	2	4	0	3
Jumps (of 1 each) required	5	8	5	4	5	4	6

Next, draw lines through each point on both number lines and introduce the game, *Taxicab Arithmetic.*

"Let's agree that each line we draw is a street on which taxicabs can travel. Of course, they must stay on the streets, but they can travel in any direction. The distances between intersections are *blocks.*

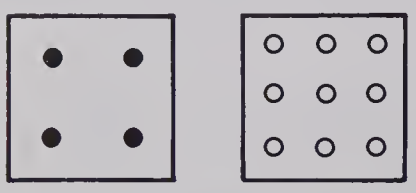


"We will call the West-East number line, W-E *Main Street,* and the South-North number line, S-N *Main Street.* Suppose a taxicab starts at 2 on S-N Main Street; it travels four blocks east and two blocks south. Where does it end up? How many blocks did it travel?" (It ended up at 4 on W-E Main Street, and it went six blocks.) "Could it have gotten there by a shorter route?" (No) "If it had stayed on

one of the Main Streets all the way, would it have traveled more blocks?" (No)

Try many trips from points on one Main Street to points on the other. Always, the shortest way is the same as if the trips were made only on the Main Streets. This game develops the two-dimensional concept that enables us to use the same diagram as a suggestion of multiplication. "Select any corner or intersection *not* on a Main Street. Draw a line from that point to take the shortest trip to each Main Street. How many square blocks are surrounded by those two trips and the two Main Streets?"

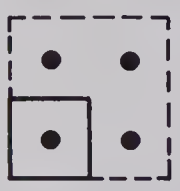
q. Square arrays. Sometimes it is possible to arrange things in a square array. For example:



In the above pictures, four dots have been arranged in two columns and two rows; and nine dots have been arranged in three columns and three rows.

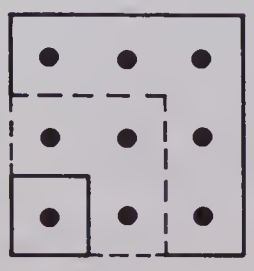
The following pictures:

suggest it is possible to arrange 0 things in 0 rows, and one thing in one row. The question is, "What numbers of things can be arranged in this way?" (0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, etc.) One way to suggest how this particular series of numbers grows is to begin with one row and one column.

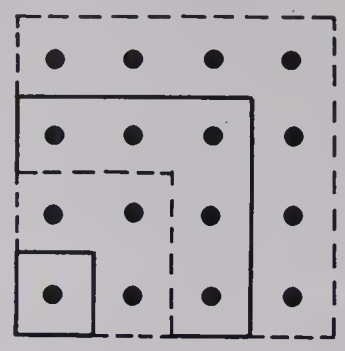


Then arrange two rows and two columns of dots, for example:

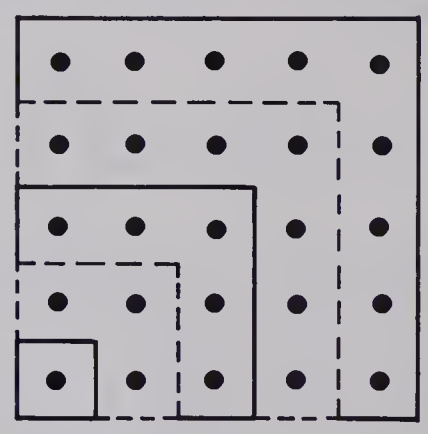
To one dot we have added three dots . . . and $1 + 3 = 4$. Next make three rows and three columns of dots:



To one and three dots we have added five dots . . . and $1 + 3 + 5 = 9$. Continue by making four rows and four columns of dots.



To one and three and five dots we have added seven dots . . . and $1 + 3 + 5 + 7 = 16$. Next:



To one and three and five and seven dots, we have added nine dots . . . and $1 + 3 + 5 + 7 + 9 = 25$. The above series of examples result in a pattern that can be easily shown in a table. Since order in addition does not change the results, reverse the order followed above to make the pattern more obvious:

$$1 = 1$$
$$3 + 1 = 4$$
$$5 + 3 + 1 = 9$$
$$7 + 5 + 3 + 1 = 16$$
$$9 + 7 + 5 + 3 + 1 = 25$$

"How can we extend this pattern?"

$$11 + 9 + 7 + 5 + 3 + 1 = \_\_\_\_ \ (36)$$
$$13 + 11 + 9 + 7 + 5 + 3 + 1 = \_\_\_\_ \ (49)$$
$$15 + 13 + 11 + 9 + 7 + 5 + 3 + 1 = \_\_\_\_ \ (64)$$

"How far can the pattern be extended?

r. Factoring and prime numbers. If a sum or a product is given, we can always find at least one pair of addends or factors, respectively, and, sometimes, many pairs of addends or factors. This can lead us to an investigation of how many pairs of whole numbers can be the factors for specific numbers.

Every number can be the product of itself and one, taken as a pair of factors. In every case, except $1 \times 1 = 1$, this particular pair of factors will be two different numbers. "Let's study several examples and find all the pairs of facts there are for each."

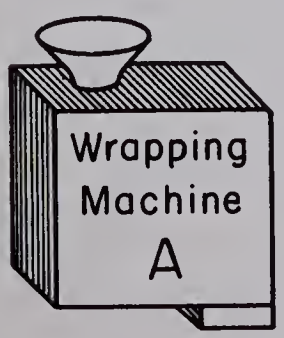
Number taken as a product	Itself and "1"	Other pairs
0	(0,1) (1,0)	(0,7) (0,99) etc.
1	(1,1)	none
2	(2,1) (1,2)	none
3	(3,1) (1,3)	none
4	(4,1)(1,4)	(2,2)
5	(5,1) (1,5)	none
6	(6,1) (1,6)	(2,3) (3,2)
7	(7,1) (1,7)	none
8	(8,1) (1,8)	(2,4) (4,2)
9	(9,1) (1,9)	(3,3)
10	(10,1) (1,10)	(2,5) (5,2)
11	(11,1) (1,11)	none
12	(12,1) (1,12)	(2,6) (6,2) (3,4) (4,3)

Continue this chart as far as time permits, and keep a separate list of all numbers that have two *different* factors, itself and *one*, but for which there are *no other pairs*. (Use of the word *different* is deliberate, because we want to exclude *one* from this list. It will be more useful to us later, if we agree that *one* is not to be considered as a prime number.) The list of numbers having *no other pairs* is:

2, 3, 5, 7, 11, 13, 17, 19, etc.

We have met this sequence of numbers before. We call them *prime numbers* because as products there is only one pair of two different factors that name them: themselves and one. Are all *prime numbers* also odd numbers? (No, there is one exception: 2.)

s. Wrapping machine arithmetic. For this activity, you first make a sketch of a crude machine on the board, labeling it "Wrapping Machine A." Any sketch will do, as long as it has a "place to put things in" and a "delivery station" or output.



	Output	
Input	Number of packages	Contents of each package
6 marbles	2	3 marbles
18 cookies		

"We don't know what's inside this machine. We only know what we put in the machine and what comes out of it. If we put six marbles in through the funnel, the machine gives us two packages, neatly wrapped. When we open the packages, we find that each contains three marbles, what kind of machine is this? Let's try it again. We put in 18 cookies. What do you suppose the machine delivers?" Of course, it might wrap them in packages of three each. Or, it might wrap them in two packages.

"Actually, it gives us two neatly wrapped packages. How many cookies do you suppose we'll find in each package?" Nine, of course.

"Now, let's see if we know what the machine will do in the following situations:

	Output	
Input	Number of packages	Contents of each package
6 marbles	2	3 marbles
18 cookies	2	cookies
30 pennies		
a dozen eggs		
ten trucks and four cars		
a half-dozen apples		
two dimes and six pennies		
three nickels and five pennies		
		7 pencils
		25 stories
		11 letters of the alphabet

The fourth item may cause a feeling of doubt. Some pupils may think the machine couldn't handle such a problem any better than it could handle one marble . . . or one nickel. But most pupils will agree that the machine would give us two packages with six eggs in a package.

With ten trucks and four cars as the problem, the machine might wear a puzzled look—if it had one to wear. The machine might put two cars and five

trucks in each package. That would be consistent. Or, it might put seven vehicles in each package: one containing four trucks and three cars; the other containing six trucks and one car.

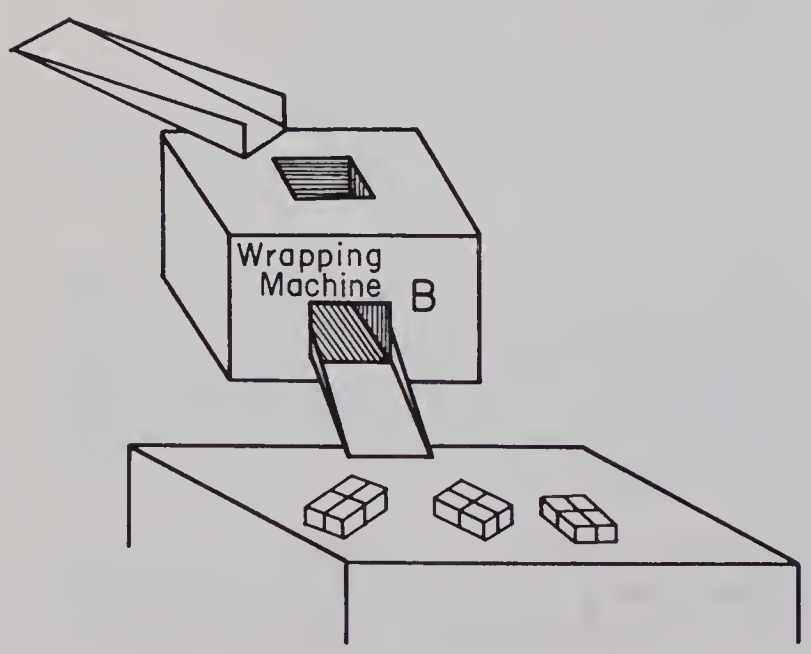
Next item—easy.

Next item—"two dimes and six pennies." Does the machine recognize that a dime is worth 10 pennies? Well, in one package there were four pennies. "What do you suppose was in the other?" Two dimes and two pennies. There were four coins in each package. Evidently, the machine simply counts the objects, and separates them into two groups with the same number in each group.

"Suppose we know the contents of one of the packages. Can we know anything about the 'input'?" If one package contains seven pencils, then the input must have been 14 objects, seven of which were pencils. It is possible that there were 14 pencils, but it is also possible that there were 11 pencils and three erasers, etc.

"What could we feed into the machine that would stump it completely?" Certainly one quarter (25c), one egg, three eggs, etc.

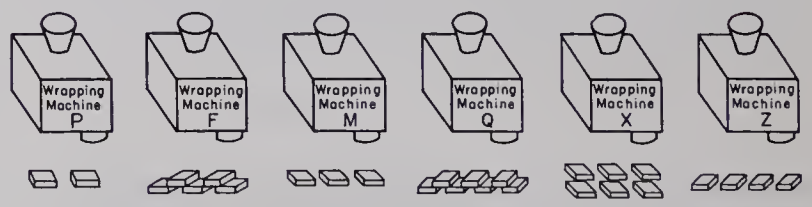
Draw another wrapping machine, "Machine B," that is identical in operation, only it always delivers three packages. If you were to feed it six marbles, you would get them wrapped neatly in three packages, with two marbles in each. Have pupils keep a record of output and input. "How many of the examples of input, that we have considered could both machines handle?" (Six marbles, 30 pennies, a dozen eggs . . . but not 10 trucks and four cars, nor two dimes and six pennies.)



	Output	
Input	Number of Packages	Number items in each package
6 marbles	3	2 marbles
30 pennies	3	10 pennies
1 doz. eggs	3	4 eggs
a set of tricycle wheels	3	1 wheel
1 penny, 4 nickels, 2 dimes, 5 quarters		4 coins
2 doz. eggs		
		7 marbles
		4 nickels, 2 pennies
		3 trucks, 3 cars
		11 stars

This same activity can be extended with machines that deliver four, five, six, etc., packages with the same number of items in each package.

After the idea has been sufficiently developed, draw a series of machines and a series of input-output records on the board. "There has been a mix-up, and the tables (or records) have been separated from the machines they refer to. Can you match each machine with its record?"



Machine M		
Input	No. Pkg.	Items ea. Pkg.
12 coins		
		8 eggs
	3	2 whistles
30 books		
		7 pencils

Machine Q		
Input	No. Pkg.	Items ea. Pkg.
14 marbles		
		10 beads
35 cookies		5 cookies
21 cars		

Machine F		
Input	No. Pkg.	Items ea. Pkg.
10 wheels		
5 apples		
15 cents		
40 stamps		
		4 pears
		$\frac{1}{2}$ doz. eggs
		10 marbles

Machine P		
Input	No. Pkg.	Items
		7 books
		3 rulers
8 pears		
3 doz. bananas		18 bananas

Machine X		
Input	No. Pkg.	Items
1 doz. eggs		a pair of eggs
12 trucks		
		5 books
		10 erasers
		1 apple, 3 oranges

Machine Z		
Input	No. Pkg.	Items
6 prs. shoes		3 shoes
2 doz. Oranges		
		$\frac{1}{2}$ doz. eggs
		7 marbles

Of course, the first record must belong to "Wrapping Machine M"—because the number of packages is shown.

The clue for the second machine is given by the information that it wraps 35 cookies in a certain number of packages—and there are five cookies in each package . . . so, there must be seven packages—and that chart must belong with Machine Q.

Similar clues are given, with varying degrees of directness, for all others except the record that belongs with Machine F. This can be solved in at least two ways: (a) since all other records have been assigned to a machine, the unassigned record and machine must belong together; or (b) assuming the machine can successfully handle all the inputs listed in the record, it must wrap each in five packages because the machines that wrap 2, 3, 4, 6, or 7 packages could not handle all the inputs successfully.

Next, we have enough information now to complete each line of each record. And, we could expand each record as far as we like.

To summarize these results, another chart can be drawn:

	Machine P		Machine M	Machine Z	Machine F	Machine X	Machine Q
Number of Items in Input	In Pkg.	Each	Pkg.	Pkg.	Pkg.	Pkg.	Pkg.
2	2	1					
3			3 1				
4	2	2		4 1			
5					5 1		
6	2	3	3 2			6 1	
7							7 1
8	2	4		4 2			
9			3 3				

Should the chart be extended through inputs of 20 or more, interesting patterns will be detectable. In this chart, if a machine cannot handle a particular "input," the space pertaining to it is to be crossed out. If the class enjoys the "wrapping machine," continue with the idea by asking about this "can't handle" situation. "Machine M wraps every input in three packages. If you fed seven marbles into it, the machine would be stumped. Could a machine be designed to solve this problem some way?" Consider every answer as a possibility. Would the output be confusing? One possible answer would be to have a machine designed so that it would mark the third package, or "only one" or, "There's only one—a full box would be three." This answer anticipates "division remainders," but our main attention is still focused on the idea of multiplication and real life situations where it has application.

Each episode of this "wrapping machine" story is as open-ended as you want to make it. Each fact of multiplication can be introduced at least once and repeated as often as required. Individual work can be easily assigned. "Make up a chart of your own that might accompany Machine X that has an output of six packages. List 10 inputs and their appropriate outputs. List 10 inputs the machine couldn't handle." The problems can be applied to the other machines.

21. Pairs of answers

A question such as, *How many are 3 and 5?* that can be answered off-hand with a number is usually a trivial question and appropriate only for drill. A question such as, *How many are 357 and 294?* may test the pupil's ability to *regroup* or to *carry,* but the answer again ends with a number: 651.

"Pairs of Answers" is an activity that requires

thinking, but it does not involve large numbers. A notation that may be new for many children is used in asking a question that involves "pairs of answers." For example:

This is an open sentence. It is neither true nor false. The box and triangle are to be considered as *reserved seats* for pairs of whole numbers. If "1" approaches the box or square, and if its partner is "3," they can both sit down because 1+3 = 4. Another pair that would be permitted to take these reserved seats is "0" and "4." In fact, any pair can be seated, provided their sum is 4. A record of pairs with a sum of 4 can be made as follows:

	A	B	C	D	E	F
□	1	0	3			
△	3	4	1			

The pairs mentioned above have been recorded under A and B in the chart. Under C another pair has been recorded. Are there more? Yes, "4" in the square and "0" in the triangle; also, the pair composed of "2" and "2." *That is five pairs. Are there more?* No!

This activity does not involve large numbers, but the questions it raises are not trivial. Pupils are given an opportunity to record the suitable pairs in a pattern that makes an eloquent answer to the question, *Are there any more pairs?*

0 and 4
1 and 3
2 and 2
3 and 1
4 and 0

22. Chain reactions

Given a rule and a pair of numbers, it is possible to set off a *chain reaction* that can be continued across the page, and on indefinitely.

Sample rule: "Each number is to be the sum of the two numbers immediately to the left of it. Given:

1 2 ___ ___ ___ ___ ___ ___ ___ ___

the next term must be 3, because 1 + 2 = 3; the next term is 5, because 2 + 3 = 5; the next is 8, because 3 + 5 = 8; etc. Thus, this chain reaction gives us:

1 2 3 5 8 13 21 34 55 89

and it could keep on going. Change the given numbers, and another chain reaction is under way.

A chain reaction can be worked in both directions. Let the rule remain the same, but change the given numbers and place them mid-point of the reaction:

___ ___ ___ ___ 10 16 ___ ___ ___ ___ ___ ___

Under the rule, the number to the left of 10 must be 6, because 6+10 = 16; the number to the right of 16 must be 26, because 10+16 = 26. So, extended in both directions, the results are:

2 2 4 6 10 16 26 42 68 110 ___ ___

Another sample rule: *Each number must be twice as large as the one to its left.* Now it takes only one given number to start the action.

1	2	4	8	16	32	64
Given	(2×1)	(2×2)	(2×4)	(2×8)		
128	256	512	___	___	___	___	___

Another sample rule: *Each number must be the difference of the two numbers above it: the one on its left subtracted from the one on its right. The bottom row is always the same number.*

1 ___ ___ ___ ___ ___
2 ___ ___ ___ ___ ___
3 ___ ___ ___ ___ ___

Told that the bottom row is all alike, it is possible to begin the reaction:

1 A ___ ___ ___ ___
2 B ___ ___ ___ ___
3 3 3 ___ ___ ___

If 2 in the second line must be gotten by subtracting 1 from A (or A−1 = 2), then A must be 3, because 3−1 = 2. By the same reasoning, B−2 = 3, so B must be 5 because 5−3 = 2. Completed, the chain reaction would look like this:

1 3 8 16 27 41 58 78 101 127
2 5 8 11 14 17 20 23 26
3 3 3 3 3 3 3 3

These chain reactions get underway with a mini-

mum of direction and *go someplace.* They require accuracy, since each in turn depends on its predecessor and any little error along the way will make every succeeding term wrong. Children get a real feeling of satisfaction in being able to demonstrate this accuracy.

A final word

The *Math Workshop for Children* has not invented a new mathematics! Nor have its authors spun novel theories of education. Rather, in it they have tried to bring into a whole the most universally accepted ideas of both educators and mathematicians.

Basically, then, the *Math Workshop for Children* takes hold of traditional subject matter and prescribes for its study large doses of solid mathematics, rather than the traditional presentation of isolated facts.

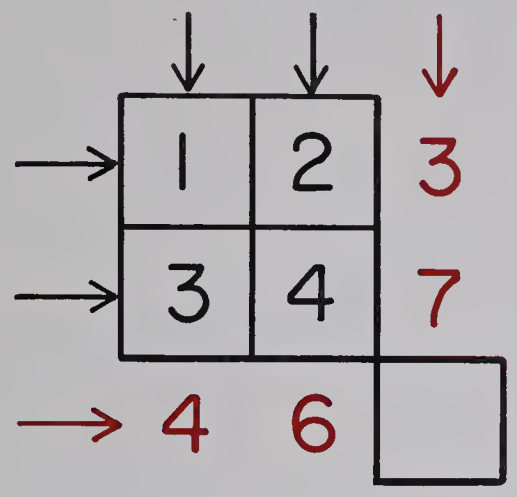
We have accepted the challenge of mathematicians and educators to think about the problem long enough and with the creative interest it takes to distill mathematical ideas until their essence emerges in the simplicity that new young minds insist upon—and respond to.

Robert W. Wirtz
Morton Botel
W. W. Sawyer

Notes to accompany *Work Text* pages 1-5

Each of the first nine pages of *Work Text D* considers some variation of the Cross Number Puzzle, a form designed by the authors of the *Math Workshop for Children*. The Cross Number Puzzle makes clear the important elements of mathematical structure, not by *name*, but *in fact*.

Below is the most typical addition-subtraction form of the Cross Number Puzzle, a simple example. The addition is carried out as indicated by the arrows:



We add across the first two rows, then down the right column, and record the results in the check box. Next we add down the two columns, then across the bottom, and our result is already written in the check box.

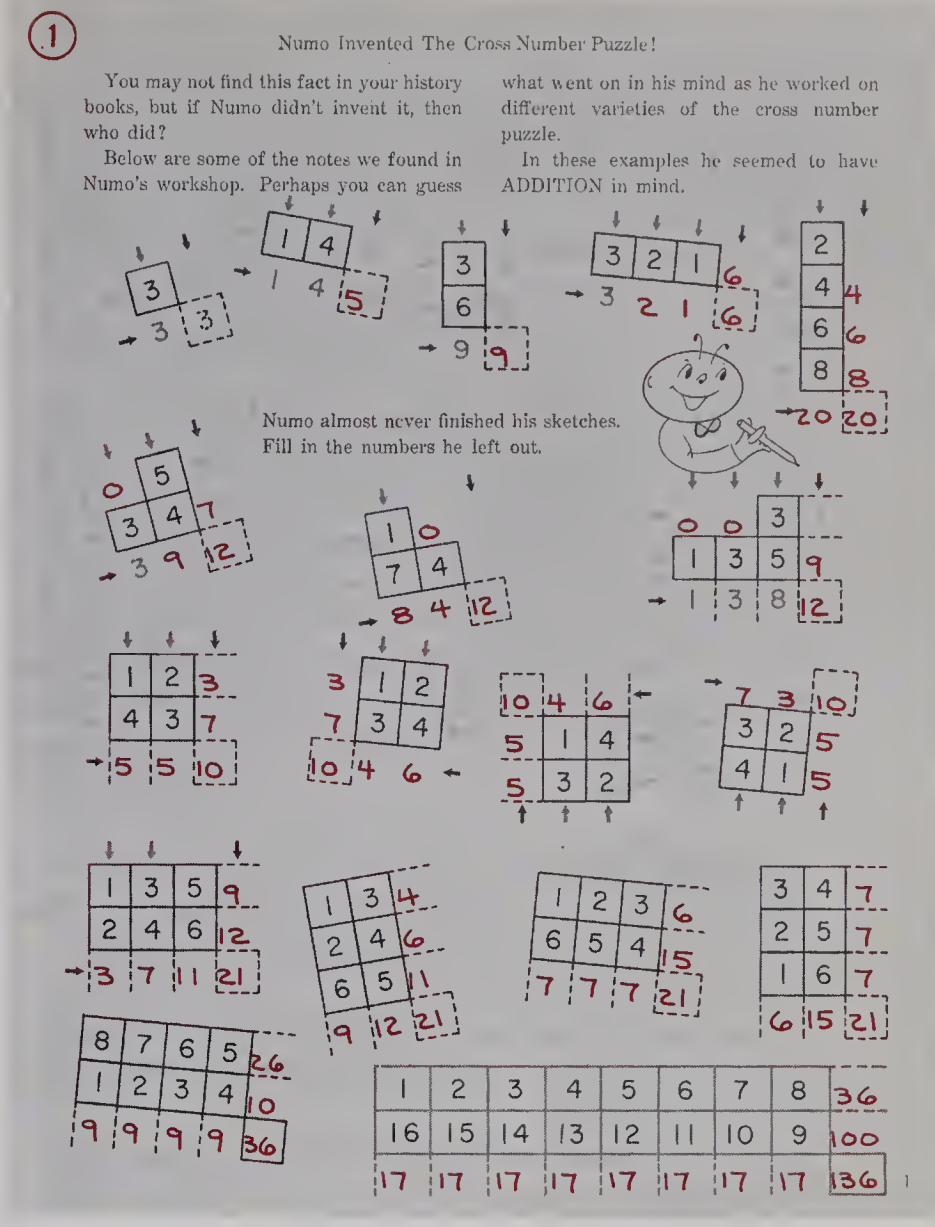
You can start discussion by asking, "Will this always happen? Does it depend on the four numbers we write in the four beginning cells—1, 2, 3, and 4 in our example? Does the order in which they are written have an influence on the outcome?"

These are questions that bear investigation—unless, of course, the children already know the secret. Let everyone try many, many examples until he is quite convinced that he connot select any group of four numbers that will not give the same surprising result. Nor does arrangement have any bearing on the outcome.

The In-Any-Order Rule

Why does this happen? Simply because we have an example of a fundamental principal of addition—the order in which we perform addition is immaterial.

$$1 + 2 = 2 + 1$$
$$1 + 2 + 3 = 3 + 2 + 1 = 2 + 1 + 3$$
$$(1 + 2) + (3 + 4) = 3 + 7 = 10$$
$$(1 + 3) + (4 + 2) = 4 + 6 = 10$$
$$(3 + 2) + (1 + 4) = 5 + 5 = 10$$


1
Numo Invented The Cross Number Puzzle!
You may not find this fact in your history books, but if Numo didn't invent it, then who did?
Below are some of the notes we found in Numo's workshop. Perhaps you can guess what went on in his mind as he worked on different varieties of the cross number puzzle.
In these examples he seemed to have ADDITION in mind.
Numo almost never finished his sketches. Fill in the numbers he left out.

Seen in a different context, this is more obvious:

1	4	2	2
2	3	3	4
3	2	1	1
+4	+1	+4	+3
10	10	10	10

And, of course, the basic principle holds no matter what group of numbers is under consideration. That many people recognize this principle in one context but not in another, underlines the necessity for us to present every idea in a multiplicity of settings so that children do not confuse the idea with its background nor with the shorthand or words we use to express it.

Reconstructing the puzzle

The box on the bottom right can be called a "check box," because the sum of the column above it must always be the same as the sum of the row to its left. If the four numbers given are placed in the original block of four boxes, the pupil is to proceed to the check box by a series of six additions. If any of the four given numbers is written outside the original set of boxes, the pupil must use both subtraction and addition in order to complete the puzzle.

Examples A through I below have been carefully

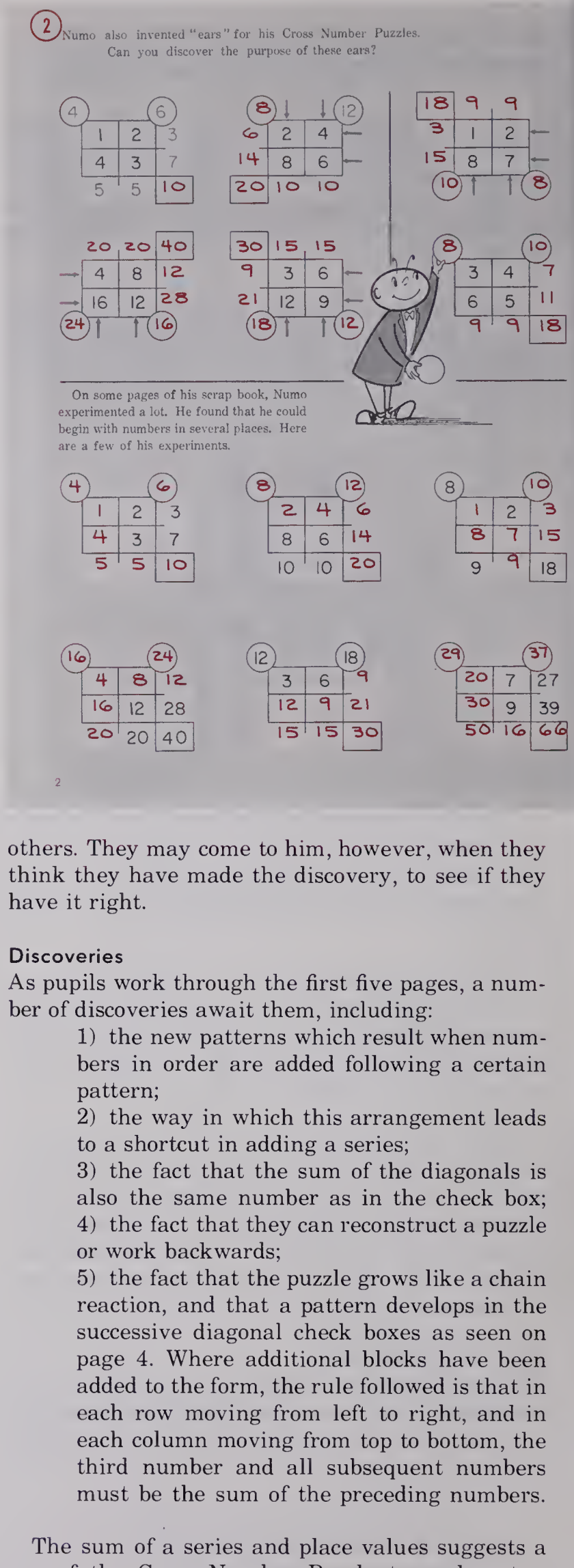
selected to lead into the "carrying" algorism in addition. Although none of the examples in the top row suggest the problem of "carrying," they are planned to be of help when the second row is tackled. D may be read as asking, "What is the sum of 15 + 6?" In working through the puzzle it becomes clear that 15 + 6 = 10 + 11. "What is the sum of 10 + 11?"

In the third row, we have moved ahead to more difficult examples. The first puzzle asks, in part, "How much is 55 - 27?" and thus involves the idea of *borrowing*. But the examples are so designed that in working back through the puzzle, the child will be doing the equivalent of borrowing and he will be able to read a correct answer without any help or even hints. He may even discover on his own the idea of borrowing, but probably won't feel the need for giving it any name.

The final three sets of boxes on page B call for a change in the rules. In the first 3 rows addition was used. Now subtraction is to be used.

In example L the pupil is to reconstruct the puzzle, or lead into the set of four heavily outlined blocks by using addition.

As pupils experiment with the Cross Number puzzles, see how soon they learn the secret. Make certain they keep the secret so that other pupils may discover for themselves whether it always works and why. Anytime in the program a pupil learns a secret, he becomes an *expert,* but he must not tell



2

others. They may come to him, however, when they think they have made the discovery, to see if they have it right.

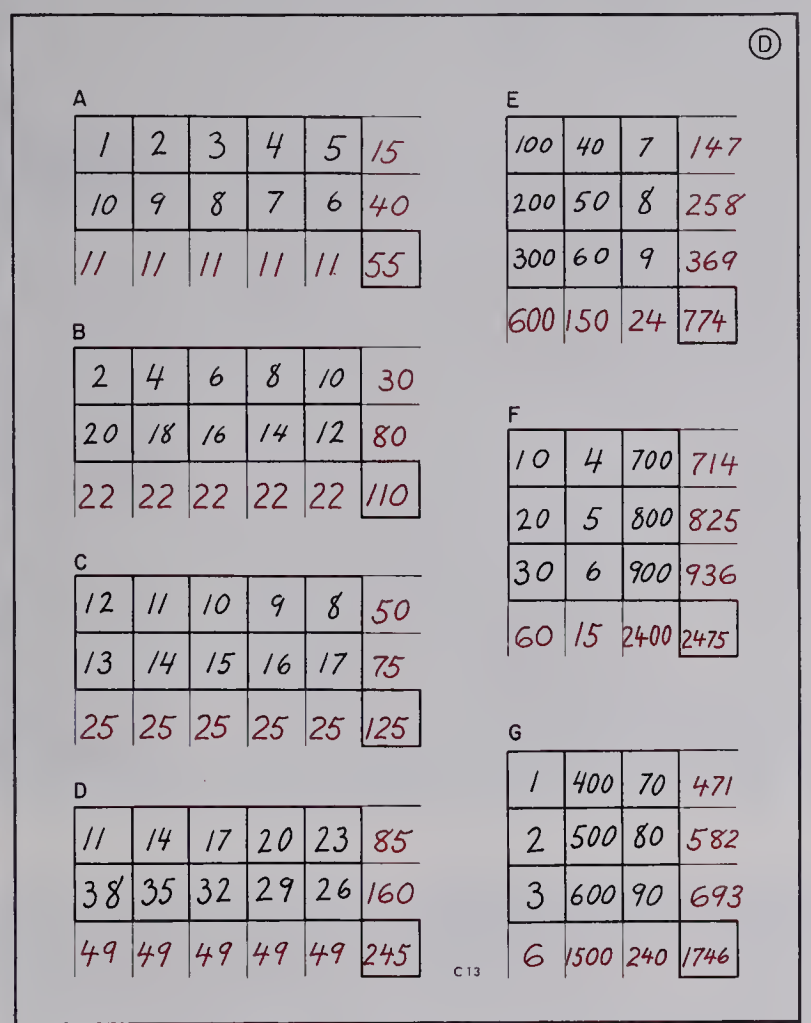
Discoveries

As pupils work through the first five pages, a number of discoveries await them, including:

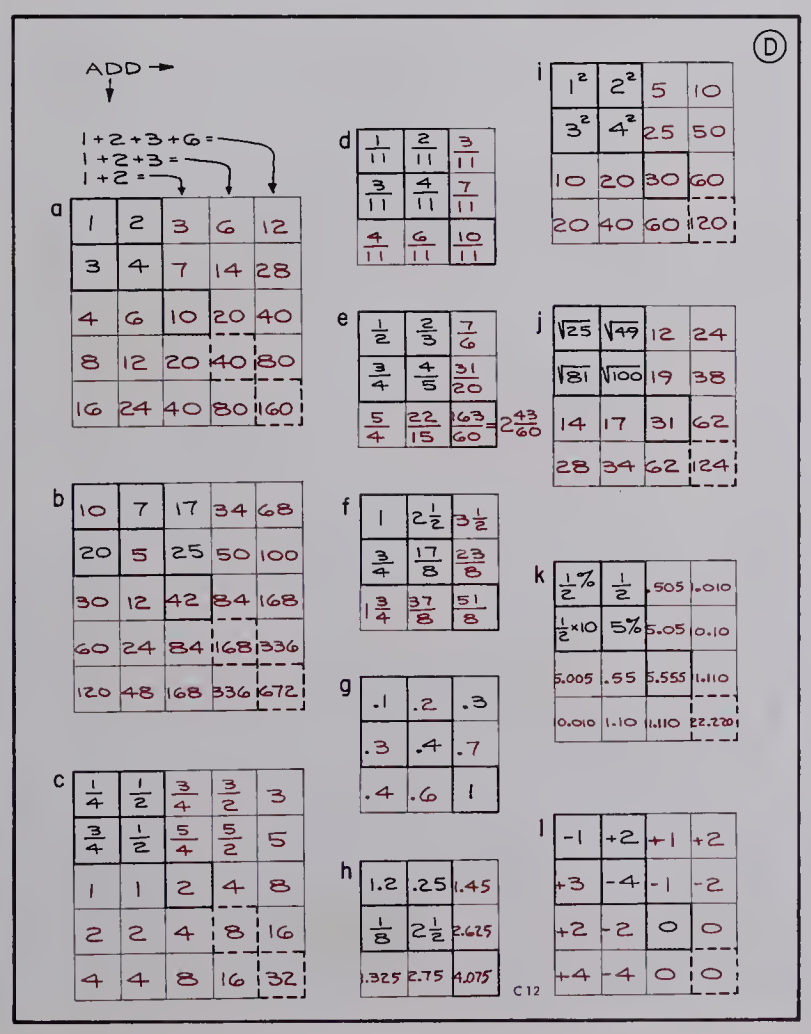
1) the new patterns which result when numbers in order are added following a certain pattern;
2) the way in which this arrangement leads to a shortcut in adding a series;
3) the fact that the sum of the diagonals is also the same number as in the check box;
4) the fact that they can reconstruct a puzzle or work backwards;
5) the fact that the puzzle grows like a chain reaction, and that a pattern develops in the successive diagonal check boxes as seen on page 4. Where additional blocks have been added to the form, the rule followed is that in each row moving from left to right, and in each column moving from top to bottom, the third number and all subsequent numbers must be the sum of the preceding numbers.

The sum of a series and place values suggests a use of the Cross Number Puzzle to explore two widely separated ideas. First, it can help find short-

cuts for the sum of a series—in these examples, *constant difference* series in which each term is a specified number more or less than its predecessor. Second, the puzzles can help develop a deeper understanding of the algorism for addition involving *carrying*. A certain pattern in the three examples helps focus attention on the importance of *place value*.



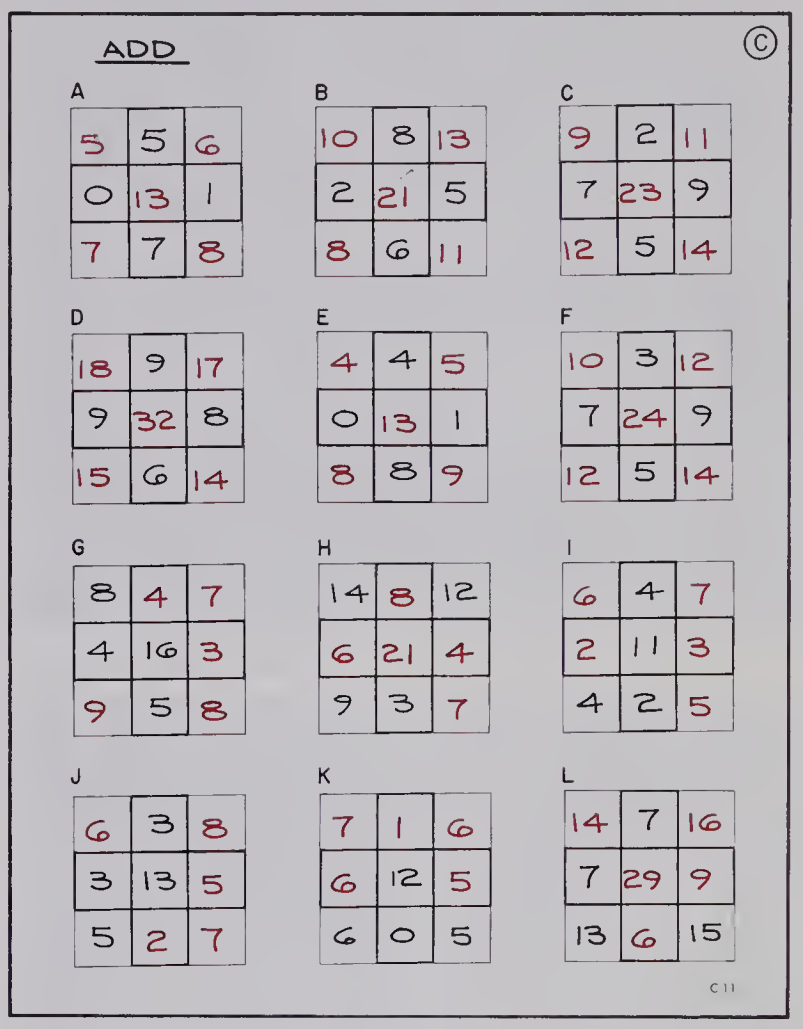
Providing for individual differences



The form of the Cross Number Puzzle on page 4 is adaptable to many levels of difficulty as suggested by the range of examples given at the bottom of the previous column.

Another form of the Cross Number Puzzle

The illustration below shows another version of the Cross Number Puzzle. The rule here is that the number in each corner is the sum of the numbers adjacent to it. The center box is the check box—it must be the sum of the numbers in the corners of each diagonal.



The Cross Number Puzzle and multiplication

Finally, go back to the first version of the Cross Number Puzzle used in the examples of addition (and subtraction). Now, however, change the rules to multiplication (and division).

The first example in the illustration below shows successive multiplications—and the *check box* seems to function as it did when the rule was "add." Does it always work? In example B, the same numbers are rearranged, and in example C, one of them is changed. It still works. "What if we change more than one of the numbers?" Now the pupils are off again into independent research.

Example G initiates a study of 0 as a factor in any chain of multiplication. In I, only one factor is given—0. This factor determines the three zeroes shown in red—and the lettered cells can accomodate any numbers.

3

14		14
10	4	14
10	4	14
20	8	28

28		21
20	1	21
20	8	28
40	9	49

28		28
20	8	28
20	8	28
40	16	56

35		35
30	5	35
30	5	35
60	10	70

35		49
30	5	35
40	2	42
70	7	77

42		42
40	2	42
40	2	42
80	4	84

Lu[illegible]nt . . . without ears!

7	7	14
7	7	14
14	14	28

	7	7	14
7	7	7	21
7	14	14	35

7	7	7	21
7	7	7	21
14	14	14	42

	7	7	7	21
7	7	7	7	28
7	14	14	14	49

7	7	7	21
7	7	7	21
7	7	7	21
21	21	21	63

7	7	14
7	7	14
7	7	14
7	7	14
7	7	14
7	7	14
42	42	84

7	7	14
7	7	14
7	7	14
7	7	14
35	35	70

7	7	14
7	7	14
7	7	14
7	7	14
28	28	56

		7	7	7	7	7	35
7	7	7	7	7	7	42
7	14	14	14	14	14	77

4 CROSS NUMBER PUZZLES

a

10	7	17	34	68
0	5	5	10	20
10	12	22	44	88
20	24	44	88	176
40	48	88	176	352

b

6	0	6	12	24
8	20	28	56	112
14	20	34	68	136
28	40	68	136	272
56	80	136	272	544

c

30	3	33	66	132
0	9	9	18	36
30	12	42	84	168
60	24	84	168	336
120	48	168	336	672

d

5	20	25
5	0	5
10	20	30

e

5	10	15
5	10	15
10	20	30

f

10	5	15
0	7	7
10	12	22

g

10	8	18
10	6	16
20	14	34

h

3	10	13
9	20	29
12	30	42

i

5	30	35	70
6	30	36	72
11	60	71	142
22	120	142	284

j

10	4	14	28
40	8	48	96
50	12	62	124
100	24	124	248

k

10	8	18	36
10	8	18	36
20	16	36	72
40	32	72	144

l

30	6	36	72
30	6	36	72
60	12	72	144
120	24	144	288

H can be reconstructed from the single product, because to have 1 as a product, all the factors (in whole numbers) must be 1. J is determined by the three products given—2 and 2 must be the factors of 4, since using 1 and 4 creates impossibilities. K and L are determined by the check boxes.

D

Multiply!

A

1	×	4	= 4
×		×	×
3	×	2	= 6
=		=	=
3	×	8	= 24

B

1	2	2
4	3	12
4	6	24

C

1	4	4
5	2	10
5	8	40

D

1	4	4
3	2	6
3	8	24

E

1	2	2
4	3	12
4	6	24

F

1	4	4
5	2	10
5	8	40

G

4	5	20
3	0	0
12	0	0

H

1	1	1
1	1	1
1	1	1

I

a	b	c
0	d	0
0	e	0

J

2	5	10
2	5	10
4	25	100

K

7	1	7
1	7	7
7	7	49

L

11	1	11
1	3	3
11	3	33

C3

5

How many is the sum of all the whole numbers 1 through 24? After this question in Numo's notebook, he had begun the work. Can you finish it for him?

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 + 23 + 24

3 7 11 15 19 23 27 31 35 39 43 47

10 26 42 58 74 90

36 + 100 + 164 = 300

Then Numo must have put the cross number puzzle to work on the same problem. Here is what we found

A

1	2	3	4	5	6	7	8	9	10	11	12	?	78
24	23	22	21	20	19	18	17	16	15	14	13		222
25	25	25	25	25	25	25	25	25	25	25	25	300	

The problem of the green ? was tackled this way—

B

1	2	3	4	5	6	21	
12	11	10	9	8	7	57	
13	13	13	13	13	13	78	

C

13	14	15	16	17	18	93?
24	23	22	21	20	19	129
37	37	37	37	37	37	222

D

1	6	7
2	5	7
3	4	7
6	15	21

E

7	12	19
8	11	19
9	10	19
24	33	57

F

13	18	31
14	17	31
15	16	31
42	51	93

G

19	24	43
20	23	43
21	22	43
60	69	129

"Ah hah!" exclaimed Numo, "Now I can find the sum of all the whole numbers 1 through 100.

H

1	2	3	4	5	6	7		45	46	47	48	49	50	1275
100	99	98	97	96	95	94						52	51	3775
101	101	101	101	101	101			101	101	101	101	101		5,050

Notes to accompany *Work Text* pages 6-9

On the next four pages of his workbook, Numo has developed a technique for dividing a rectangular array with a horizontal or vertical line, or both, and thus creating a Cross Number Puzzle. The check box occurs in a different position sometimes, and the sum of the rows is written on the left rather than the right.

Numo wonders whether these changes in format alter in any way the workings of the puzzle. No, of course not. After all, he has really been finding the answer to the question, "How many dots in each example?" Dividing a rectangular array into smaller groups turns out to be a handy check on counting.

In fact, Numo described in his notebook the array of dots in each section of the larger array. In the first example on *Work Text* page 6, he has one group that contains 3 rows and 6 columns, another that contains 3 rows and 4 columns. To summarize:

$3 \times 6 = 18$
$3 \times 4 = 12$
$18 + 12 = 30$

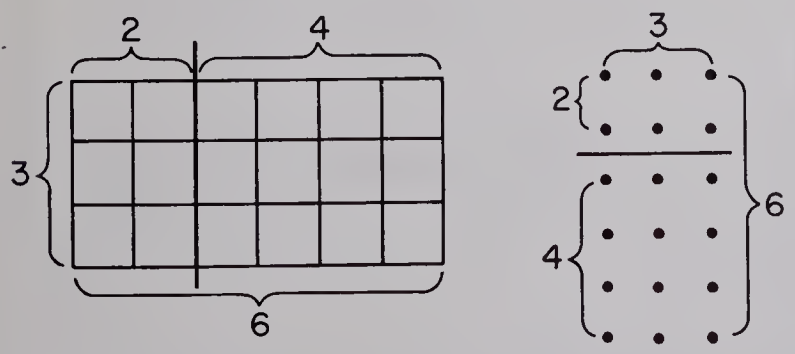
In the first example on page 7, Numo found 21 dots above the green line and 14 dots below it. There are 15 dots to the left of the blue line and 20 dots to the right of it. It might be summarized as:

3×3 = 9		3×3 + 3×4 = 21		3×(3+4) = 21
3×4 = 12	or	2×3 + 2×4 = 14	or	2×(3+4) = 14
2×3 = 6				
2×4 = 8				
9+12+6+8 = 35		21+14 = 35		21+14 = 35

Algorisms of long multiplication

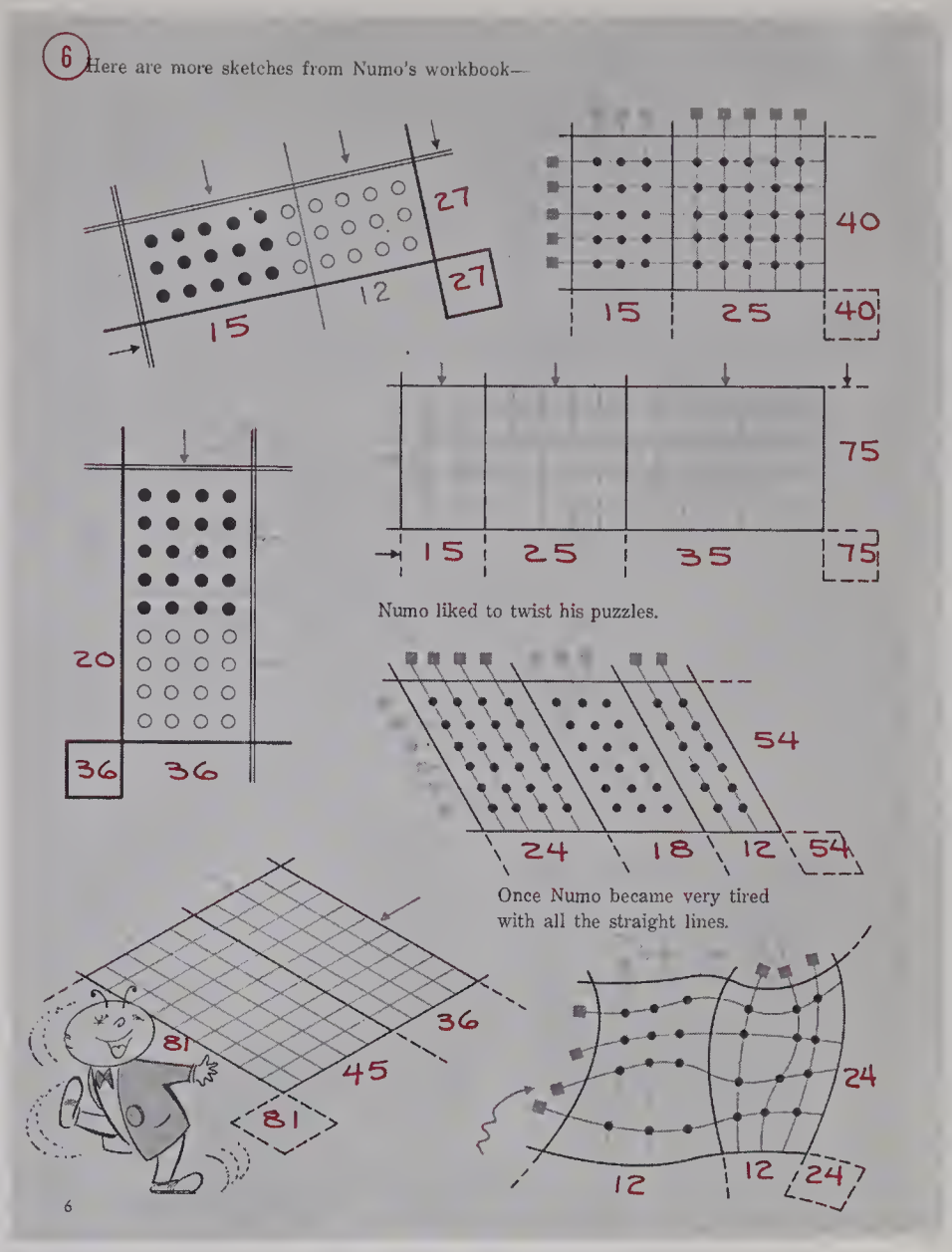
Basic to the steps or algorisms of multiplication is a principle of arithmetic that, expressed in algebra, is abbreviated: $a \times (b+c) = (a \times b) + (a \times c)$.

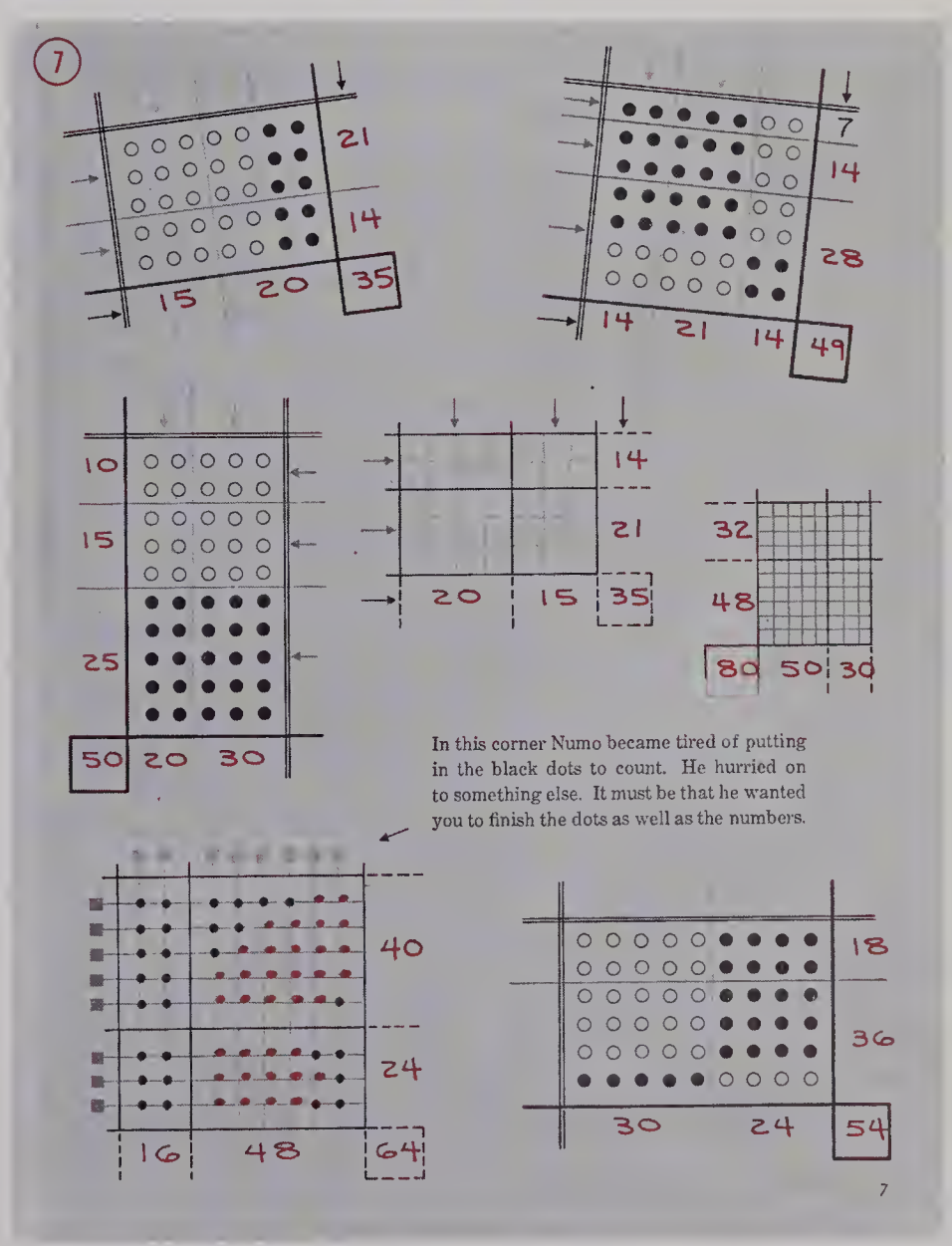
The idea is suggested by the following:

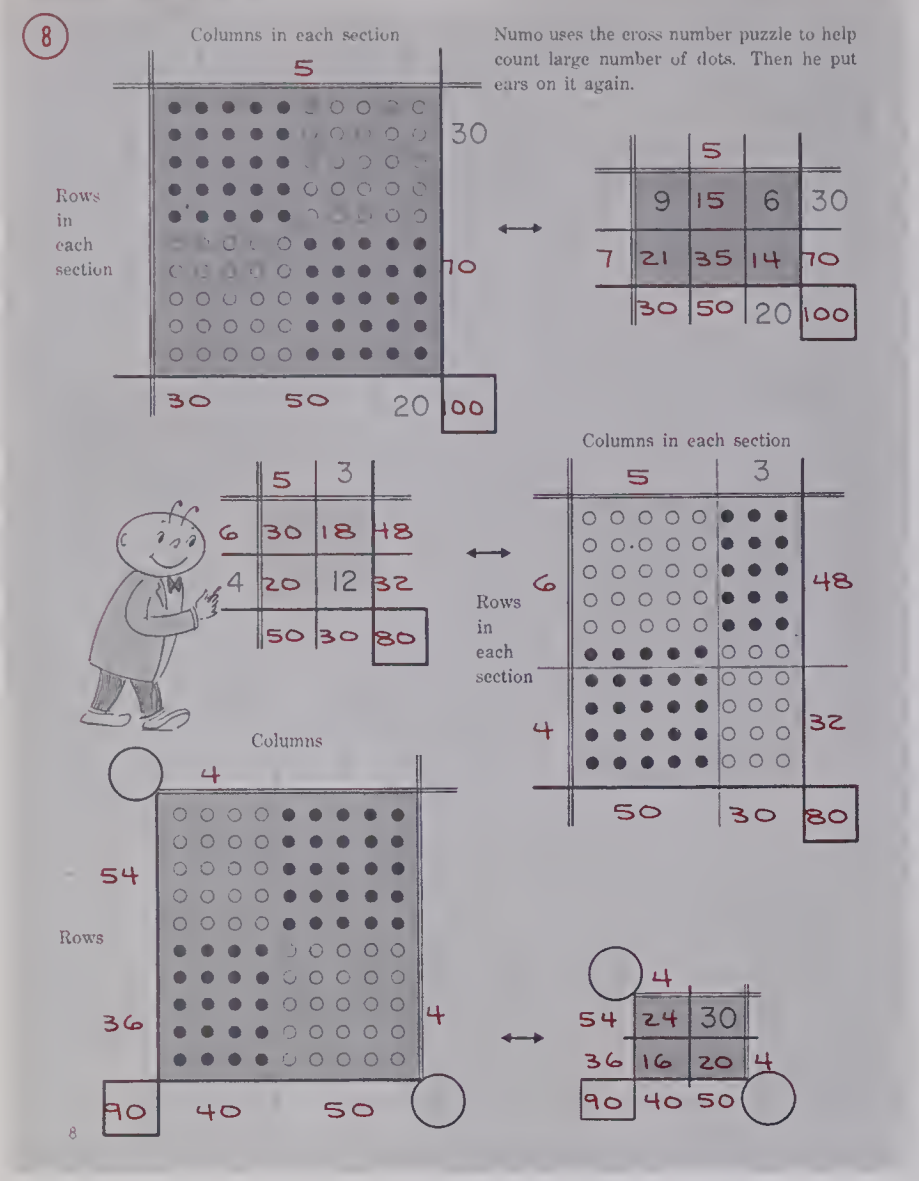


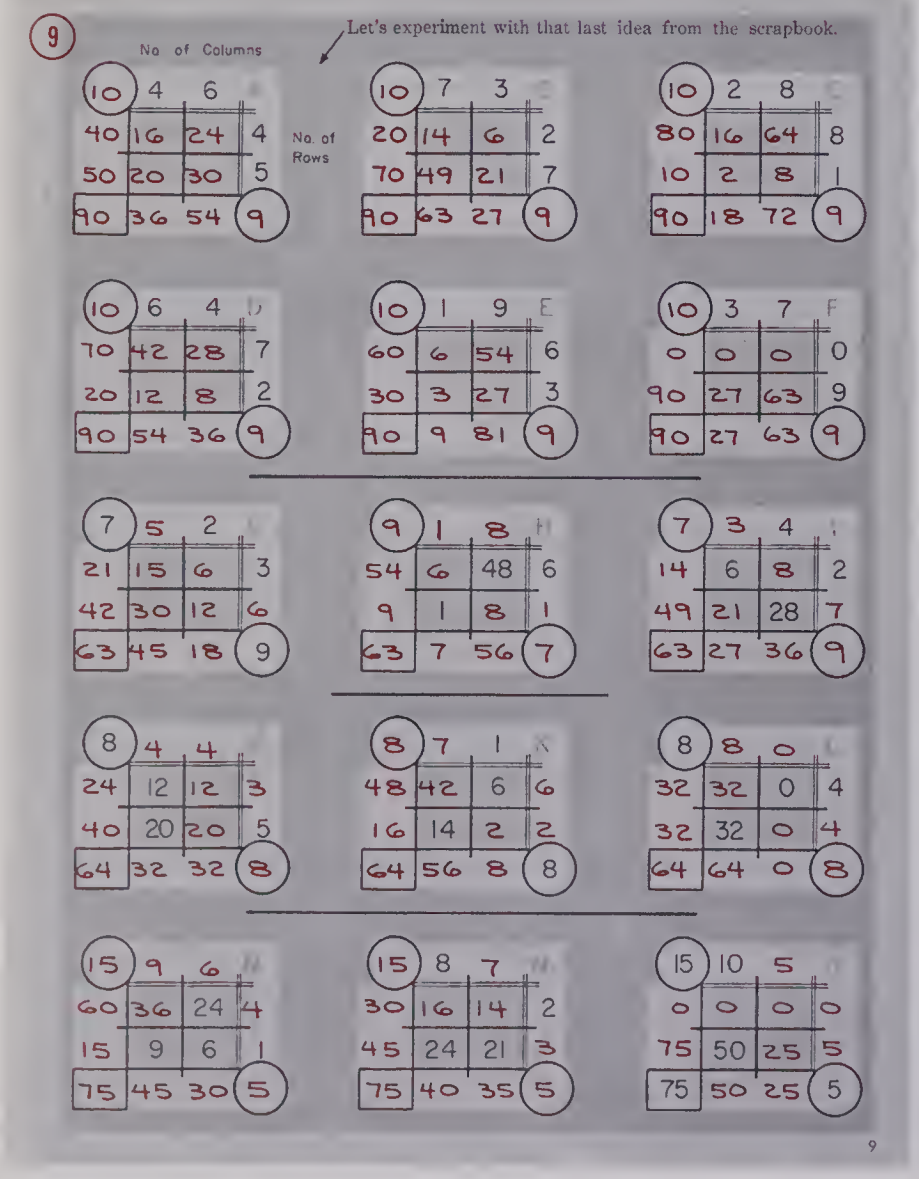
Both pictures suggest that:

$3 \times 6 = 3 \times (2+4) = (3 \times 2) + (3 \times 4) = 6 + 12 = 18$






8
Columns in each section
Numo uses the cross number puzzle to help count large number of dots. Then he put ears on it again.
Rows in each section
Columns in each section
Rows in each section
Columns
Rows
8



A few examples of the same idea are given below:

$5 \times 11 = 5 \times (1+10) = (5\times1) + (5\times10) = 5 + 50 = 55$

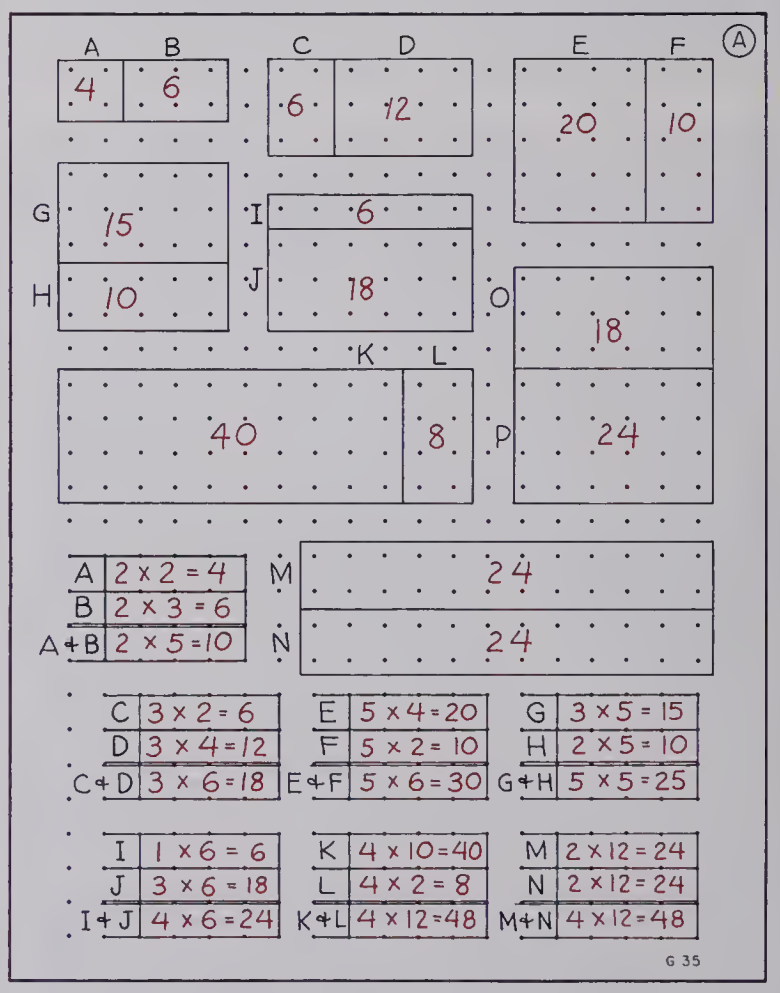
$9 \times 15 = 9 \times (5+10) = (9\times5) + (9\times10) = 45 + 90 = 135$

$4 \times 38 = 4 \times (8+30) = (4\times8) + (4\times30) = 32 + 120 = 152$

Notice that to use the traditional algorism *with understanding*, each of the steps outlined above is taken in one form or another.

Multiply:	11	15	38
	5	9	4

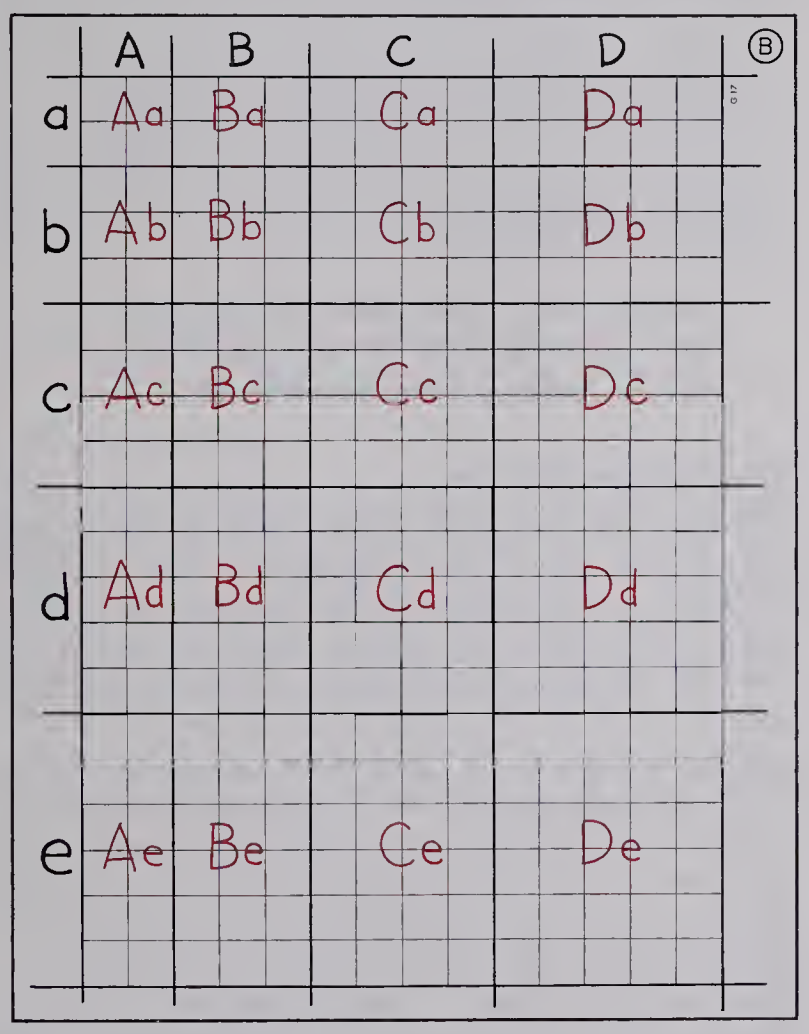
The sample page given below is an introduction to this basic principle of arithmetic. The urge to have children put this idea into words should be postponed for a long time. The rule can be discovered by most children and used effectively if the experience of discovery is not destroyed by premature verbalization.



The pictures above are meant to suggest many examples of multiplication. Example A suggests that $2 \times 2 = 4$; B suggests that $2 \times 3 = 6$; A and B together suggest that $2 \times 5 = 10$.

The multiplication suggested by each separate picture and by each pair is recorded in the lettered boxes at the bottom of the page. In recording the facts suggested by C, by D, and by C and D, we illustrate a way in which children often record the results in a form that does not lead as directly as we would like to a discovery. However, this part of the experience need not be talked about for some time.

The following picture illustrates a version of coordinate labelling—a row-column shorthand. Any of the boxes on the page can be located by the use of a capital letter and a small letter: a *letter pair* or pair of letters. It should be noted that it does not matter in what order the letters are put down: capital and small, or small and capital. In answering a question such as "How many rectangles contain five fewer squares than Dd?" the answer would be Cd (or dC) and cD (or Dc). Many other such questions can be asked.



The examples on page 9 make use of a Cross Number Puzzle in which the four central blocks record the products of the factors at the top of each column and at the right end of each row. First, each row and column is added. Then these sums are added, and the new sum is written in the check box. After examples A through F have been completed, record, in the top left-circle, the sum of the factors at the top; and record, in the bottom right-circle, the sum of the factors at the right.

This breaking up of factors into subfactors, then cross-multiplying them to come up with numbers to be added, is exactly the procedure that is followed in such multiplication as 37×89: $(30+7) \times (80+9) = (7\times9+7\times80) + (30\times9+30\times80) = (63+560) + (270+2400) = 3293$. Children can acquire a feeling for the rationale of this process without rote memorization as they work slowly through these and other pages. They will learn to work through increasingly difficult problems without the help, yet, of *algorisms or rules of procedure.*

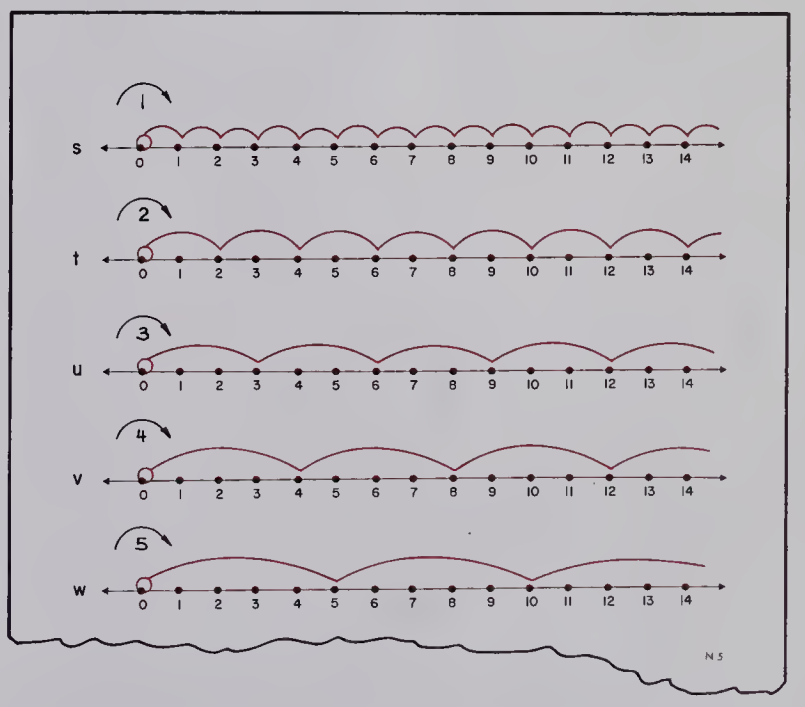
On page 9, the first six examples suggest that $10 \times 9 = (4+6) \times (4+5) = (2+8) \times (8+1) = (6+4) \times (7+2) = (1+9) \times (6+3) = (3+7) \times (0+9)$. Each factor, 10 and 9, can be changed to any of its other names in the form of ____+____. Each of the other rows of examples explores this important fact further.

Work Text pages 6 through 9 are deepening an understanding of regrouping that is essential in comprehending the important steps of long multiplication and long division.

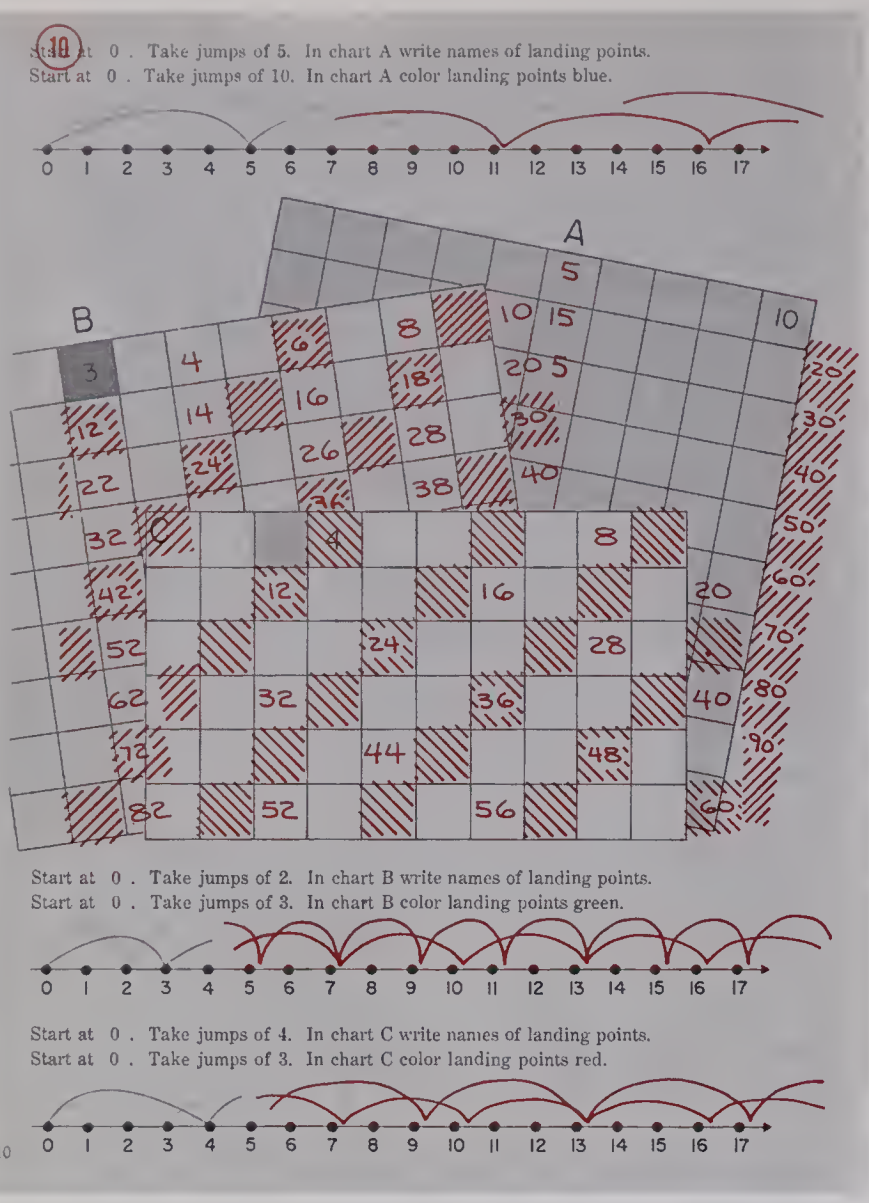
Notes to accompany *Work Text* pages 10-12

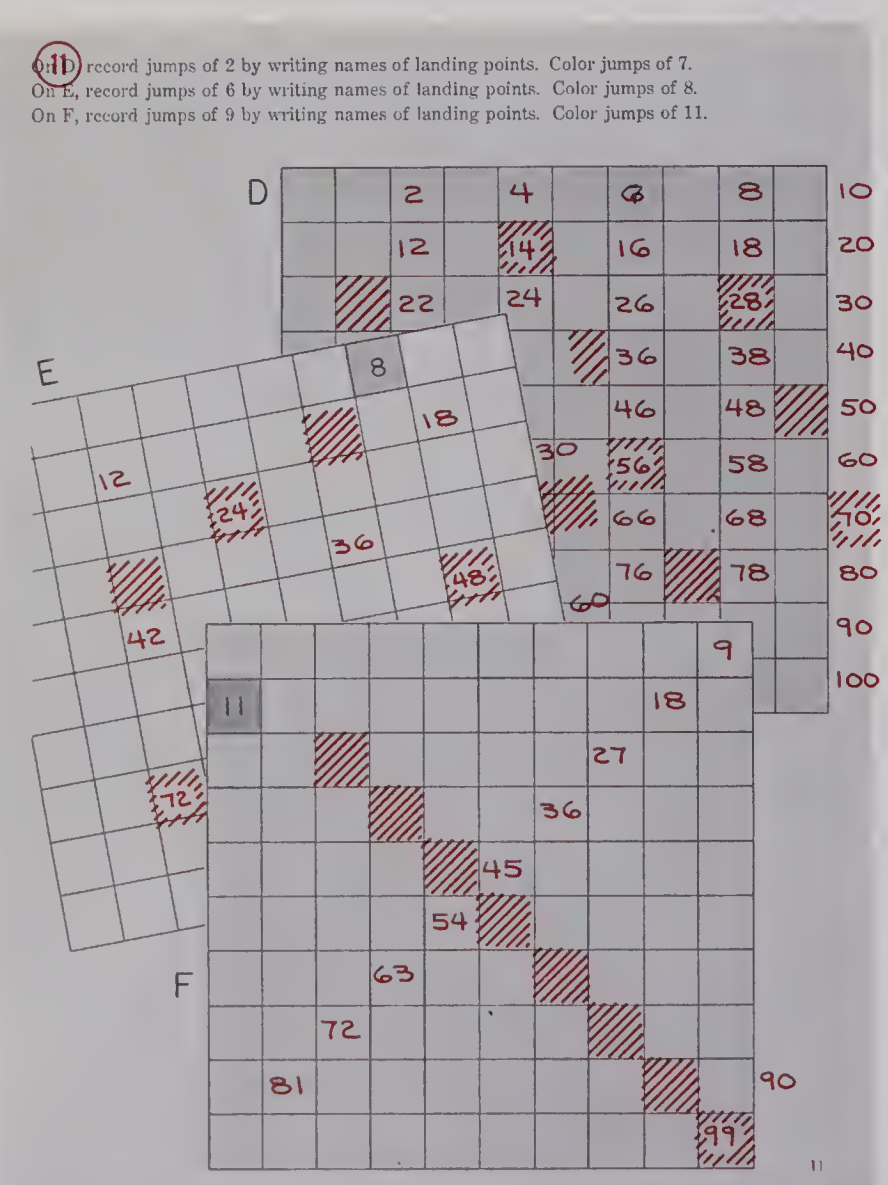
The idea explored on pages 10 through 12 is what Professor W. W. Sawyer calls "Wallpaper Arithmetic."

What happens when we start at "O" on a number line and take successive jumps of a given amount? The teacher can begin this activity on the blackboard or by using *Work Form* N5, as below:



In addition to using number line forms, the class can also record the results of successive jumps on wallpaper forms, coloring the successive starting and landing points.


10
Start at 0 . Take jumps of 5. In chart A write names of landing points.
Start at 0 . Take jumps of 10. In chart A color landing points blue.
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
A
B
C
Start at 0 . Take jumps of 2. In chart B write names of landing points.
Start at 0 . Take jumps of 3. In chart B color landing points green.
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
Start at 0 . Take jumps of 4. In chart C write names of landing points.
Start at 0 . Take jumps of 3. In chart C color landing points red.
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17


11
On D, record jumps of 2 by writing names of landing points. Color jumps of 7.
On E, record jumps of 6 by writing names of landing points. Color jumps of 8.
On F, record jumps of 9 by writing names of landing points. Color jumps of 11.
D
E
F
11

Different groups of the class can be assigned the creation of a set of these records. Mounted on tag board or tacked on a bulletin board—arranged in order—they become a conversation piece.

Overlapping patterns

On page 10, pupils are asked to write the names of the landing points when taking jumps of 5, and to color the landing points representing jumps of 10 in blue. Of course, half the jumps of 5 coincide with jumps of 10.

A coincidence of part of the pattern formed by each pair of numbers occurs in each of the six wallpaper forms on pages 10 and 11. Once the pattern is discovered, some pupils will be able to determine the next place the numbers will overlap.

This pattern, of course, is nothing more than a graphic picture of the fact that the number of the size of the jump is one of the factors of the number on which it lands. So:

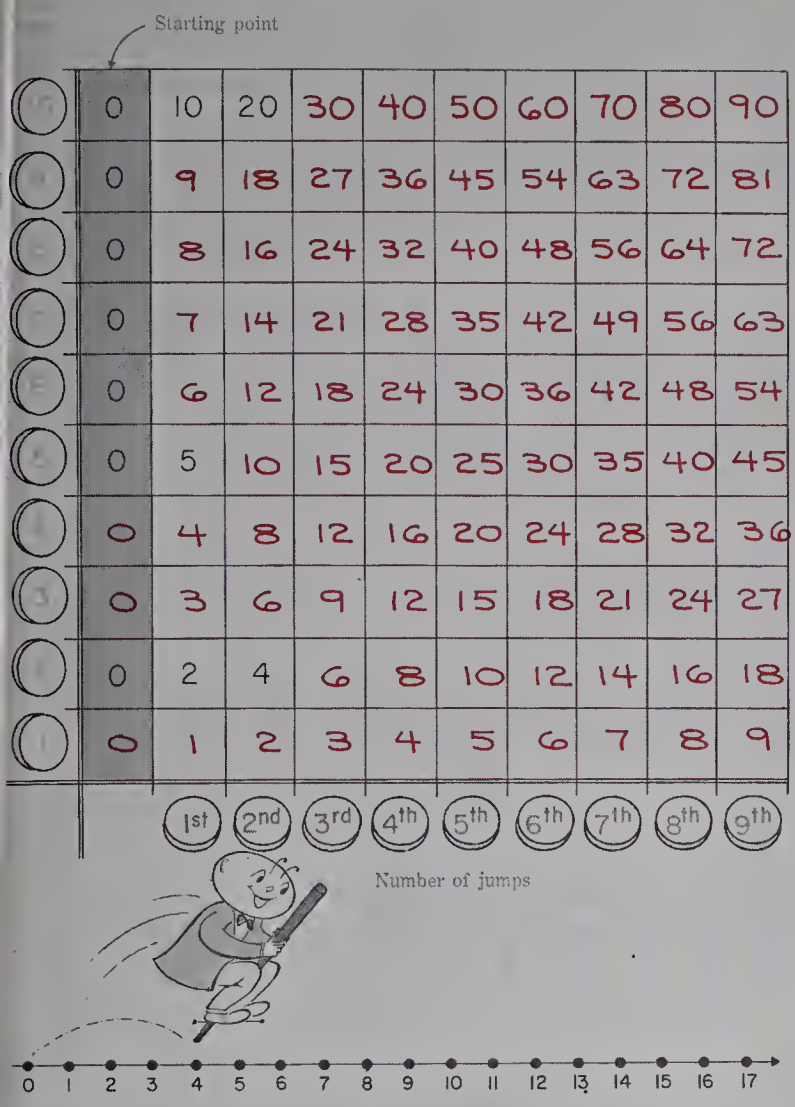
In A—5 is one of the factors of 10, 20, 30
In B—2 and 3 are factors of 6, 12, 18
In C—4 and 3 are factors of 12, 24, 36
In D—2 and 7 are factors of 14, 28
In E—6 and 8 are factors of 24, 48
In F—9 and 11 are factors of 99

A multiplying machine

A useful way to record these results is on a single grid. Pupils are asked to do this on *Work Text* 12. Each row in the grid lists the starting and landing points of the first nine jumps. The keys on the side indicate the size of each jump. After this grid has been filled in by the class, it can be used to answer such questions as: "If I took seven equal jumps, and my first landing point was 6, where am I now?" or, "I'm on 14 after two jumps; where will my next jump take me?" Or, "I am on 49; how did I get here?" (There is only one possible answer.) Or, "I'm on 35; how many ways is it possible to get to 35?" . . . "And what are they?" These questions, of course, lead the class into the idea of division—and, properly, as the inverse of multiplication.

When such questions are no longer challenging, the teacher can turn the pupil's attention to a search for patterns in this new grid—which later will be called the *multiplying machine*. Patterns abound—simple ones like the rows of zeroes and sequences of the natural numbers; more complicated ones like the main diagonal of square numbers—1, 4, 9, 16, 25, etc. The whole grid is symmetrical

On *Work Text* page 11:



around this main diagonal (because $6 \times 7 = 7 \times 6$; $2 \times 9 = 9 \times 2$, etc.).

Further exploration

Pupils may want to make similar explorations of the coincidence of other jumps such as 2 and 6 on one sketch, 4 and 5 on another, etc., using blue for one pattern and yellow for the other pattern. One result of the approach is that the overlapping blocks come out green, a stimulating discovery.

Some pupils may want to project a series on a chart in such form as:

Jumps of:

2	2	4	6	8	10	12	...	40	42	44	46	... 48
3	3	6	9		12				42	45		48

Or some may wish to explore such numbers as 6, 10, 24, 36, 50 to see how many jump patterns coincide on these numbers.

Numbers	Jump patterns which touch
6	1, 2, 3
10	1, 2, 5
24	1, 2, 3, 4, 6, 8, 12
25	1, 5, 10, 15, 20, 25
36	1, 2, 3, 4, 6, 9, 12, 18

Notes to accompany *Work Text* pages 13-19

All algorisms (including carrying, borrowing, long multiplication and long division) depend on what we call regrouping, or the idea that "numbers have many names. Because of the decimal system we use, we can easily write any number less than 100 as a sum of a multiple of 10 and a number less than 10.

$$73 = 70 + 3;\ 28 = 20 + 8$$

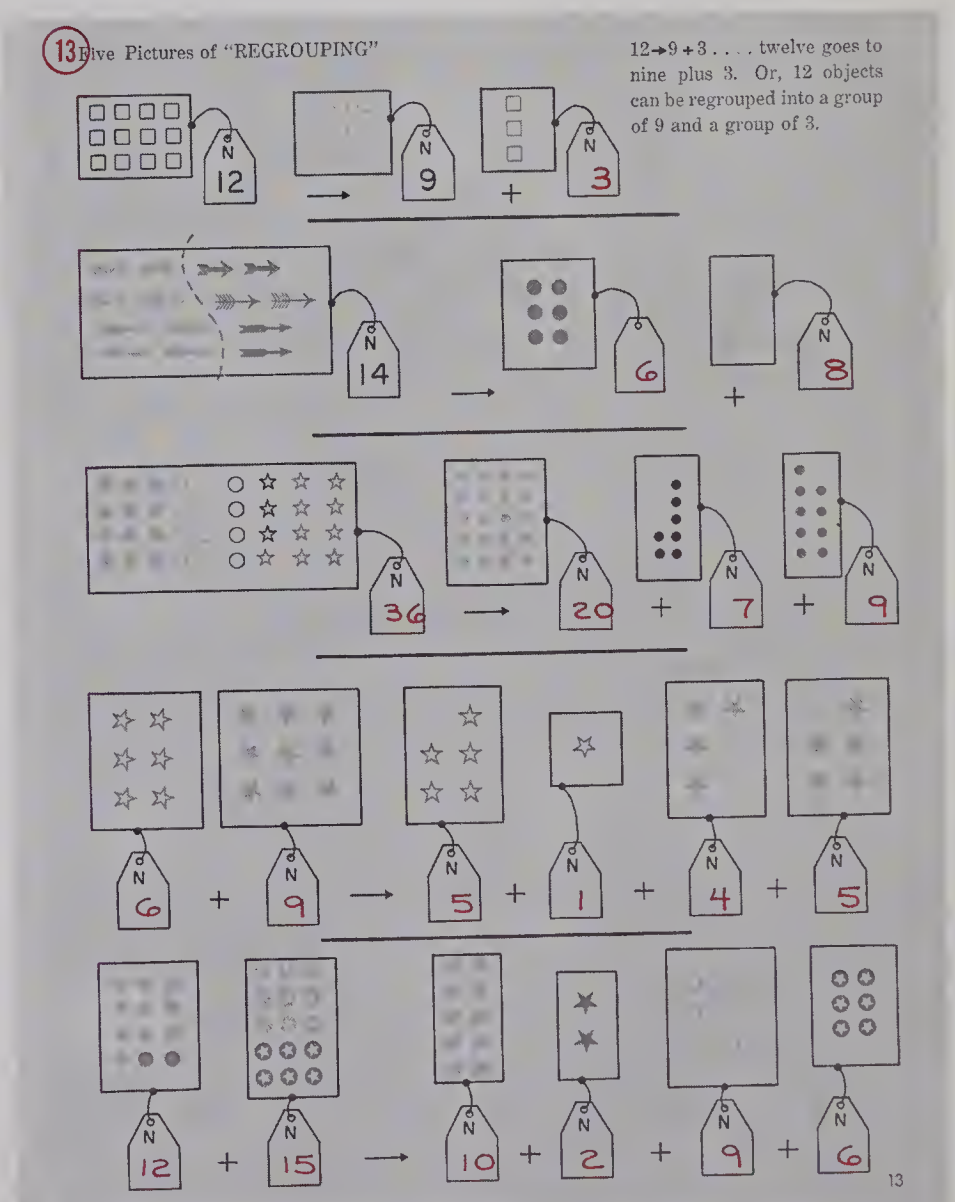
or as the sum of a multiple of 10 and a number greater than 10 and less than 20

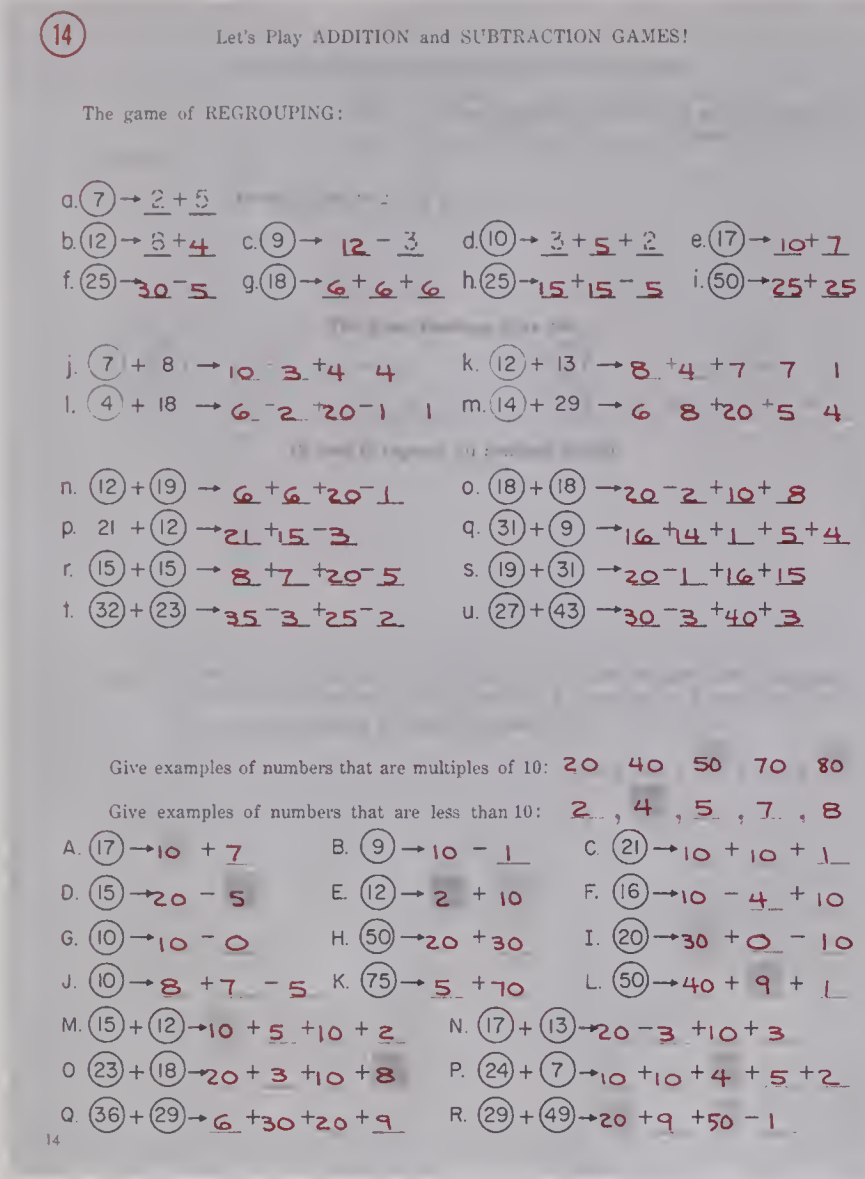
$$73 = 60 + 13;\ 28 = 10 + 18$$

And, since the order in which we add does not affect the outcome, we freely move addends around.

$$\begin{aligned} 73 + 28 &= (70+3) + (20+8) \\ &= (70+20) + (3+8) \\ &= 90 + (10+1) \\ &= (90+10) + 1 \\ &= 100 + 1 \\ &= 101 \end{aligned}$$

Whether we think of each of these steps or not,


13 Five Pictures of "REGROUPING"
12→9+3 . . . . twelve goes to nine plus 3. Or, 12 objects can be regrouped into a group of 9 and a group of 3.
N 12 → N 9 + N 3
N 14 → N 6 + N 8
N 36 → N 20 + N 7 + N 9
N 6 + N 9 → N 5 + N 1 + N 4 + N 5
N 12 + N 15 → N 10 + N 2 + N 9 + N 6
13



14 Let's Play ADDITION and SUBTRACTION GAMES!

The game of REGROUPING:

a. (7) → 2 + 5

b. (12) → 8 + 4 c. (9) → 12 − 3 d. (10) → 3 + 5 + 2 e. (17) → 10 + 7

f. (25) → 30 − 5 g. (18) → 6 + 6 + 6 h. (25) → 15 + 15 − 5 i. (50) → 25 + 25

j. (7) + 8 → 10 − 3 + 4 − 4 k. (12) + 13 → 8 + 4 + 7 − 7 − 1

l. (4) + 18 → 6 − 2 + 20 − 1 − 1 m. (14) + 29 → 6 − 8 + 20 + 5 − 4

n. (12) + (19) → 6 + 6 + 20 − 1 o. (18) + (18) → 20 − 2 + 10 + 8

p. 21 + (12) → 21 + 15 − 3 q. (31) + (9) → 16 + 14 + 1 + 5 + 4

r. (15) + (15) → 8 + 7 + 20 − 5 s. (19) + (31) → 20 − 1 + 16 + 15

t. (32) + (23) → 35 − 3 + 25 − 2 u. (27) + (43) → 30 − 3 + 40 + 3

Give examples of numbers that are multiples of 10: 20 40 50 70 80

Give examples of numbers that are less than 10: 2 , 4 , 5 , 7 , 8

A. (17) → 10 + 7 B. (9) → 10 − 1 C. (21) → 10 + 10 + 1

D. (15) → 20 − 5 E. (12) → 2 + 10 F. (16) → 10 − 4 + 10

G. (10) → 10 − 0 H. (50) → 20 + 30 I. (20) → 30 + 0 − 10

J. (10) → 8 + 7 − 5 K. (75) → 5 + 70 L. (50) → 40 + 9 + 1

M. (15) + (12) → 10 + 5 + 10 + 2 N. (17) + (13) → 20 − 3 + 10 + 3

O (23) + (18) → 20 + 3 + 10 + 8 P. (24) + (7) → 10 + 10 + 4 + 5 + 2

Q. (36) + (29) → 6 + 30 + 20 + 9 R. (29) + (49) → 20 + 9 + 50 − 1

14

we take them—with understanding or by meaningless rote. Consider:

$$\begin{array}{r} 73 \\ +\ 28 \\ \hline \end{array}$$

If we start our figuring by saying to ourselves "8 + 3 = ____," then we certainly have regrouped 73 and 28 in our minds. Otherwise, we would see no 3 and no 8 for our "8 + 3." We have "regrouped" the example to look like this:

$$\begin{array}{r} 70 + 3 \\ +\ 20 + 8 \\ \hline \end{array}$$

Next, we say to ourselves, "8 + 3 = 11; write 1 and remember 10." Again, we have regrouped: 11 = 10 + 1. We note 1 and remember to add 10 to the sum of 20 + 70 = 90 . . . 90 + 10 = 100. So, we now have this situation:

$$\begin{array}{r} 70 + 3 \\ +\ 20 + 8 \\ \hline 100 + 1 \end{array}$$

We regroup once again: 100 + 1 = 101.

The rules set forth for the regrouping games on pages 13 through 19 require that children proceed with understanding through the steps that will become the traditional algorisms for addition and subtraction. While we will speak of *regrouping* and *remembering,* and will avoid such less descriptive expressions as *carrying* or *borrowing,* each teacher ought to feel perfectly free to use the language of *carrying* and *borrowing,* if he feels secure with it.

The idea of regrouping and the use of rules in the game are introduced simply on page 13, but before proceeding with the page, consider with the class some of the ways in which 4 might be regrouped given the limitation of the following forms:

4 → ____ + ____

4 → ____ + ____ + ____

"How many ways to regroup 4 are there altogether in the above forms?"

0 + 4		4 + 0 + 0		2 + 2 + 0
1 + 3		0 + 4 + 0		2 + 0 + 2
2 + 2	or	0 + 0 + 4	or	0 + 2 + 2
3 + 1		3 + 1 + 0		2 + 1 + 1
4 + 0		3 + 0 + 1		1 + 2 + 1
		1 + 3 + 0		1 + 1 + 2
		0 + 3 + 1		
		1 + 0 + 3		
		0 + 1 + 3		

The form of the expression, as in these two examples, greatly influences the possibilities. The answer to the question, "Have we all the possibilities?" is a bit of a puzzler. And the whole activity is valuable practice in regrouping—finding other names for a given number.

The games on pages 14 through 19 will do much to help children gain an understanding of the great freedom they can employ in regrouping. Though they focus attention on arbitrary rules, they do provide the flavor of a game and the children are developing a keen sense of the many ways in which they can write a number. This understanding will help them find many shortcuts that are often more effective than the algorisms we traditionally use.

Discuss the two rules for regrouping on page 14 by exploring examples *a, j, n,* and *A* in terms of alternate forms of regrouping possible given the limitations imposed by the form.

Work Text page 15

On page 15, the game sets limitations on regrouping on the basis of the facts of arithmetic included in Table I. Thus, the result of 3 + 5 is given in Table I, but 8 + 7 is not. How can we get the result of 8 + 7? By regrouping as follows:

$8 + \underline{7} \rightarrow 8 + \underline{2 + 5} \rightarrow 10 + 5 \rightarrow 15$ or
$\underline{8} + 7 \rightarrow \underline{5 + 3} + 7 \rightarrow 5 + 10 \rightarrow 15$

The secret of this game, of course, is that addends for all sums from 0 to 10 are included in the table. But the addends for all sums from 11 to 20 must include 10 as an addend.

Work Text page 16

The basic activity on page 16 is repeated change of order of one pair of addends at a time, based on the idea we call the In-Any-Order Rule; that is, in addition, the order of addends does not change the sum.

Here is a typical puzzler: "Change 1 + 2 + 3 into 3 + 2 + 1." Here is one way a child might tackle this one:

$$1 + \underline{2 + 3} = \underline{1 + 3} + 2 = 3 + \underline{1 + 2} = 3 + 2 + 1$$

or

$$\underline{1 + 2} + 3 = 2 + \underline{1 + 3} = \underline{2 + 3} + 1 = 3 + 2 + 1$$

That solves the puzzle. Here's another problem for the front runners to explore: "How many ways are there to arrange 1, 2 and 3 in the form of ____+ ____+____?"

There are only six ways. One arrangement that strengthens the conclusion that there are only six is given below. The first number listed in each group must be 1, 2 or 3. In each case, there are only two ways to arrange the other two numbers.

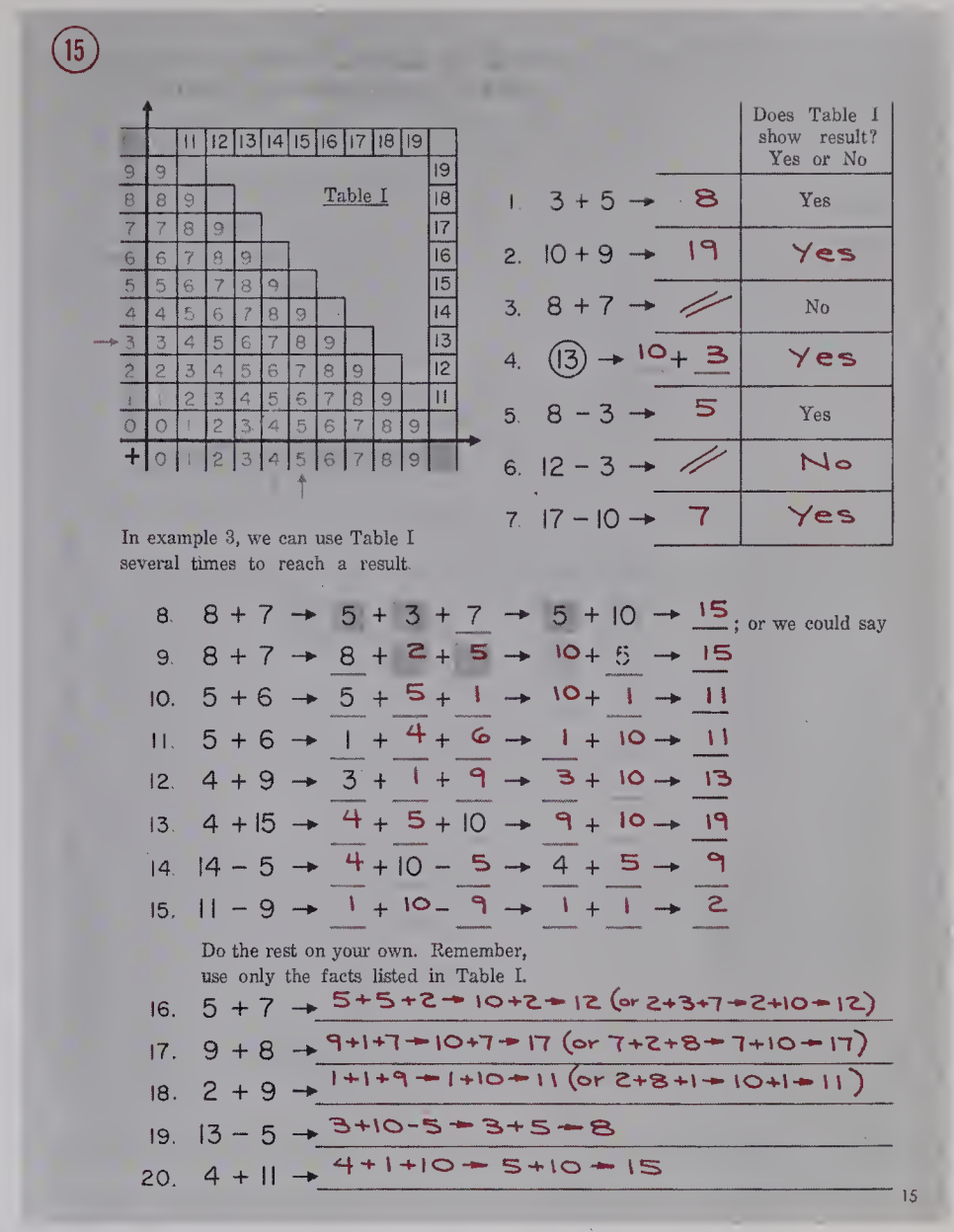
1 + ____ + ____	1 + ____ + ____
2 + ____ + ____	2 + ____ + ____
3 + ____ + ____	3 + ____ + ____

The activities on this page, in effect, ask children to find a conclusion simply by following rules and not by appealing to what they already have committed to memory nor to what appears to be obvious. And this is very much in a style that anticipates *deductive proof.* In such proofs, each step taken is justified by one of a set of established rules. No other justification is acceptable.

Work Text page 17

On page 17, the usefulness of the In-Any-Order Rule is discovered as a shortcut to addition. In the left-hand column, rearranging the number so that addends which lead to sums of 10 or multiples, often makes it possible to come up with sums quickly without writing the terms. Such experiences build real understanding of the structure of our decimal system.

In the right-hand column, the In-Any-Order Rule of addition introduces its use in subtraction.



15

Table I

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
	9	8	7	6	5	4	3	2	1	0
	19	18	17	16	15	14	13	12	11	10

		Does Table I show result? Yes or No
1.	3 + 5 → 8	Yes
2.	10 + 9 → 19	Yes
3.	8 + 7 → //	No
4.	(13) → 10 + 3	Yes
5.	8 − 3 → 5	Yes
6.	12 − 3 → //	No
7.	17 − 10 → 7	Yes

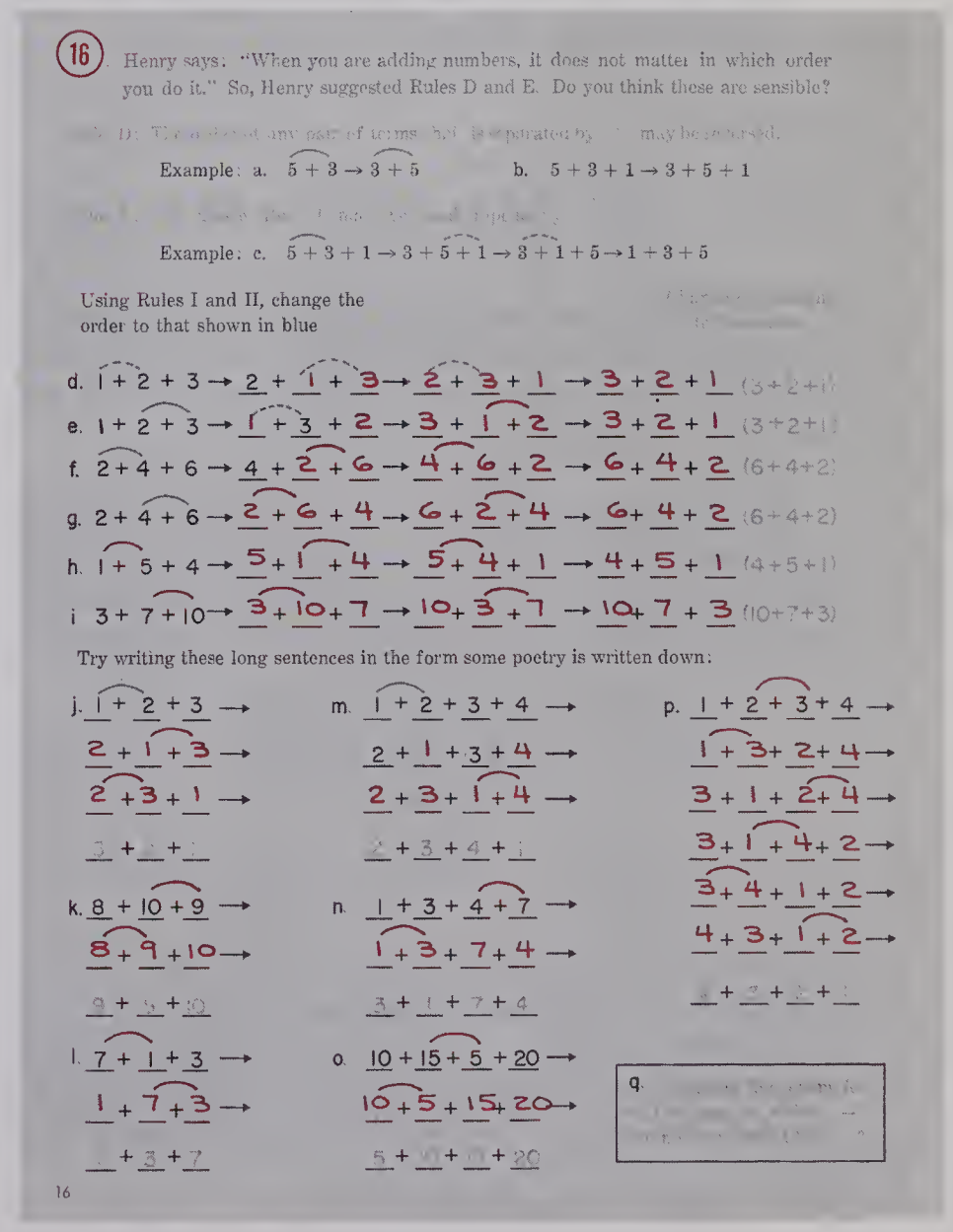
In example 3, we can use Table I several times to reach a result.

8. $8 + 7 \rightarrow 5 + 3 + 7 \rightarrow 5 + 10 \rightarrow 15$; or we could say
9. $8 + 7 \rightarrow 8 + 2 + 5 \rightarrow 10 + 5 \rightarrow 15$
10. $5 + 6 \rightarrow 5 + 5 + 1 \rightarrow 10 + 1 \rightarrow 11$
11. $5 + 6 \rightarrow 1 + 4 + 6 \rightarrow 1 + 10 \rightarrow 11$
12. $4 + 9 \rightarrow 3 + 1 + 9 \rightarrow 3 + 10 \rightarrow 13$
13. $4 + 15 \rightarrow 4 + 5 + 10 \rightarrow 9 + 10 \rightarrow 19$
14. $14 - 5 \rightarrow 4 + 10 - 5 \rightarrow 4 + 5 \rightarrow 9$
15. $11 - 9 \rightarrow 1 + 10 - 9 \rightarrow 1 + 1 \rightarrow 2$

Do the rest on your own. Remember, use only the facts listed in Table I.

16. $5 + 7 \rightarrow$ 5+5+2→10+2→12 (or 2+3+7→2+10→12)
17. $9 + 8 \rightarrow$ 9+1+7→10+7→17 (or 7+2+8→7+10→17)
18. $2 + 9 \rightarrow$ 1+1+9→1+10→11 (or 2+8+1→10+1→11)
19. $13 - 5 \rightarrow$ 3+10−5→3+5→8
20. $4 + 11 \rightarrow$ 4+1+10→5+10→15

15



16

Henry says: "When you are adding numbers, it does not matter in which order you do it." So, Henry suggested Rules D and E. Do you think these are sensible?

D: The addends of any pair of terms, but is separated by + may be reversed.

Example: a. $\overbrace{5 + 3} \rightarrow 3 + 5$ b. $5 + 3 + 1 \rightarrow 3 + 5 + 1$

E: Any pair of addends may be regrouped.

Example: c. $\overbrace{5 + 3} + 1 \rightarrow 3 + \overbrace{5 + 1} \rightarrow \overbrace{3 + 1} + 5 \rightarrow 1 + 3 + 5$

Using Rules I and II, change the order to that shown in blue

d. $\overbrace{1 + 2} + 3 \rightarrow 2 + \overbrace{1 + 3} \rightarrow \overbrace{2 + 3} + 1 \rightarrow 3 + 2 + 1$ (3+2+1)
e. $1 + \overbrace{2 + 3} \rightarrow \overbrace{1 + 3} + 2 \rightarrow 3 + \overbrace{1 + 2} \rightarrow 3 + 2 + 1$ (3+2+1)
f. $\overbrace{2 + 4} + 6 \rightarrow 4 + \overbrace{2 + 6} \rightarrow \overbrace{4 + 6} + 2 \rightarrow 6 + 4 + 2$ (6+4+2)
g. $2 + \overbrace{4 + 6} \rightarrow \overbrace{2 + 6} + 4 \rightarrow 6 + \overbrace{2 + 4} \rightarrow 6 + 4 + 2$ (6+4+2)
h. $\overbrace{1 + 5} + 4 \rightarrow 5 + \overbrace{1 + 4} \rightarrow \overbrace{5 + 4} + 1 \rightarrow 4 + 5 + 1$ (4+5+1)
i. $3 + \overbrace{7 + 10} \rightarrow \overbrace{3 + 10} + 7 \rightarrow 10 + \overbrace{3 + 7} \rightarrow 10 + 7 + 3$ (10+7+3)

Try writing these long sentences in the form some poetry is written down:

j. $\overbrace{1 + 2} + 3 \rightarrow$ $2 + \overbrace{1 + 3} \rightarrow$ $\overbrace{2 + 3} + 1 \rightarrow$ 3 + ___ + ___

k. $8 + \overbrace{10 + 9} \rightarrow$ $\overbrace{8 + 9} + 10 \rightarrow$ 9 + ___ + 10

l. $7 + \overbrace{1 + 3} \rightarrow$ $1 + \overbrace{7 + 3} \rightarrow$ ___ + 3 + 7

m. $\overbrace{1 + 2} + 3 + 4 \rightarrow$ $2 + \overbrace{1 + 3} + 4 \rightarrow$ $2 + 3 + \overbrace{1 + 4} \rightarrow$ 2 + 3 + 4 + ___

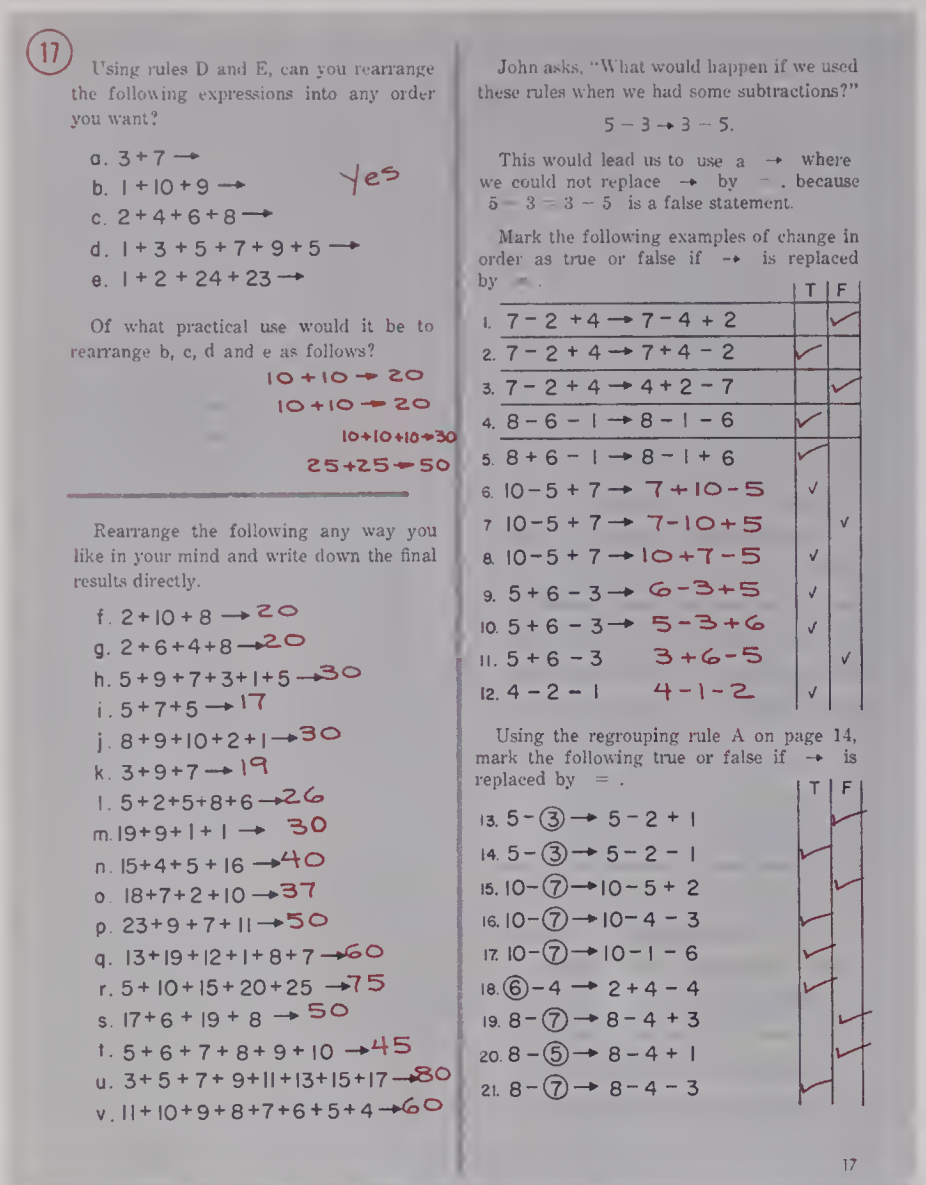
n. $1 + 3 + \overbrace{4 + 7} \rightarrow$ $1 + \overbrace{3 + 7} + 4 \rightarrow$ 3 + ___ + 7 + 4

o. $10 + \overbrace{15 + 5} + 20 \rightarrow$ $\overbrace{10 + 5} + 15 + 20 \rightarrow$ 5 + ___ + ___ + 20

p. $1 + \overbrace{2 + 3} + 4 \rightarrow$ $1 + \overbrace{3 + 2} + 4 \rightarrow$ $3 + 1 + \overbrace{2 + 4} \rightarrow$ $3 + 1 + \overbrace{4 + 2} \rightarrow$ $3 + \overbrace{4 + 1} + 2 \rightarrow$ $4 + 3 + \overbrace{1 + 2} \rightarrow$ ___ + ___ + ___ + ___

q.

16



17

Using rules D and E, can you rearrange the following expressions into any order you want?

a. 3+7 →
b. 1+10+9 → Yes
c. 2+4+6+8 →
d. 1+3+5+7+9+5 →
e. 1+2+24+23 →

Of what practical use would it be to rearrange b, c, d and e as follows?

10+10→20
10+10→20
10+10+10→30
25+25→50

Rearrange the following any way you like in your mind and write down the final results directly.

f. 2+10+8 → 20
g. 2+6+4+8 → 20
h. 5+9+7+3+1+5 → 30
i. 5+7+5 → 17
j. 8+9+10+2+1 → 30
k. 3+9+7 → 19
l. 5+2+5+8+6 → 26
m. 19+9+1+1 → 30
n. 15+4+5+16 → 40
o. 18+7+2+10 → 37
p. 23+9+7+11 → 50
q. 13+19+12+1+8+7 → 60
r. 5+10+15+20+25 → 75
s. 17+6+19+8 → 50
t. 5+6+7+8+9+10 → 45
u. 3+5+7+9+11+13+15+17 → 80
v. 11+10+9+8+7+6+5+4 → 60

John asks, "What would happen if we used these rules when we had some subtractions?"

5 − 3 → 3 − 5.

This would lead us to use a → where we could not replace → by =, because 5 − 3 = 3 − 5 is a false statement.

Mark the following examples of change in order as true or false if → is replaced by =.

		T	F
1.	7 − 2 + 4 → 7 − 4 + 2		✓
2.	7 − 2 + 4 → 7 + 4 − 2	✓	
3.	7 − 2 + 4 → 4 + 2 − 7		✓
4.	8 − 6 − 1 → 8 − 1 − 6	✓	
5.	8 + 6 − 1 → 8 − 1 + 6	✓	
6.	10 − 5 + 7 → 7 + 10 − 5	✓	
7.	10 − 5 + 7 → 7 − 10 + 5		✓
8.	10 − 5 + 7 → 10 + 7 − 5	✓	
9.	5 + 6 − 3 → 6 − 3 + 5	✓	
10.	5 + 6 − 3 → 5 − 3 + 6	✓	
11.	5 + 6 − 3 → 3 + 6 − 5		✓
12.	4 − 2 − 1 → 4 − 1 − 2	✓	

Using the regrouping rule A on page 14, mark the following true or false if → is replaced by =.

		T	F
13.	5 − ③ → 5 − 2 + 1		✓
14.	5 − ③ → 5 − 2 − 1	✓	
15.	10 − ⑦ → 10 − 5 + 2		✓
16.	10 − ⑦ → 10 − 4 − 3	✓	
17.	10 − ⑦ → 10 − 1 − 6	✓	
18.	⑥ − 4 → 2 + 4 − 4	✓	
19.	8 − ⑦ → 8 − 4 + 3		✓
20.	8 − ⑤ → 8 − 4 + 1		✓
21.	8 − ⑦ → 8 − 4 − 3	✓	

17

developed to focus upon each of these steps. Let us take several examples and discuss them, step by step.

In the example, 19 + 3 = ?, we regroup as follows:

$$\begin{array}{r} 10 + 9 \\ 0 + 3 \\ \hline 10 + 12 \end{array} \qquad \begin{array}{r} 19 \\ 3 \\ \hline ? \end{array}$$

The suggested language in this example is of the following nature: We think of 19 regrouped as 10 + 9. Think 9 + 3 = 12. In our minds, we regroup this result as 12 = 10 + 2. We record 2 and remember 10. Then, 10 + 0 = 10. This result, plus the 10 we remember, is 10 + 10 = 20. Finally, we regroup 20 + 2 as 22.

We are not concerned with the exact language, but with the argument that explains each step. This is our first casual consideration of a method that will eventually be somewhat formalized. At this point, we are primarily interested in *regrouping,* based on complete understanding of what's going on.

It is more important to provide many simple examples so that the children do not resort to manipulations, but rather think about each step they take. (Again, the teacher should feel free to talk about "carrying" and "borrowing" if "regrouping" and "remembering" seem strange.)

5 + 7 = 7 + 5, but does
6 − 7 = 7 − 5? No!

However,

5 − 2 + 8 = 5 + 8 − 2
10 + 3 − 10 = 10 − 10 + 3

Experience and reasonableness must guide the student at this time, rather than rules to be memorized. Rules at this point in the development would lead to confusion and blot out thinking.

The bottom grid on page 17 introduces a problem that may cause some disturbance. This is a problem that lies at the heart of the subtraction algorism and requires an understanding that in the example

47 − 15 = ? actually indicates that
(40+7) − (10+5) and
(40+7) − 10 − 5 or

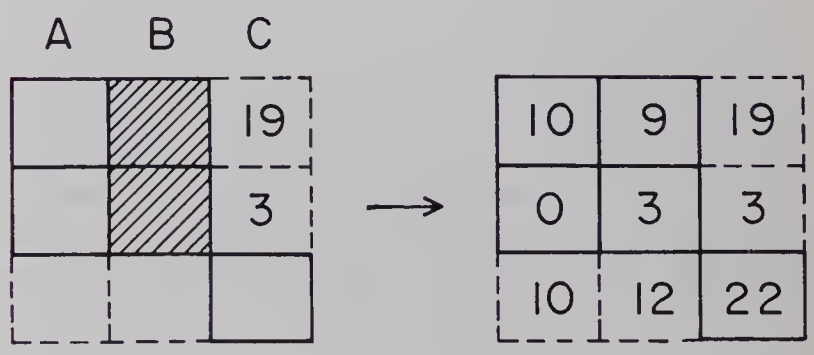
$$\begin{array}{r} 40 + 7 \\ -\ 10 - 5 \\ \hline \end{array}$$

Addition and subtraction algorisms

The rules for *Work Text* page 18 camouflage the steps required in the familiar algorisms for addition and subtraction. The activities on this page were

The Cross Number Puzzle and addition algorisms

Use the Cross Number Puzzle to help pupils see what's happening:



Column C presents the problem. The numbers are regrouped, based on the agreement that numbers in column A are reserved for multiples of 10 (including 0) and that the shaded blocks in column B are reserved for numbers less than 10.

In the subtraction problem,
47 − 15 = (40+7) − (10+5)
but are we equally sure that
(40+7) − (10+5) = 40 + 7 − 10 − 5
This is new. We purposely use the horizontal form

here because it points to an important idea we usually overlook if we consider

$$\begin{array}{r} 47 \\ -15 \\ \hline \end{array}$$

We begin by arguing that $7 - 5 = 2$. Since we have no 7 or 5 given, we must have regrouped:

$$\begin{array}{r} 40 + 7 \\ -\ 10 - 5 \\ \hline \end{array}$$

Now, is it clear that we should begin with $7 - 5 = 2$? We can ignore this problem only at the expense of a real understanding of the subtractive algorism.

"I had 47¢. I spent a dime; and then I spend a nickel. How much did I have left after I spent the dime? How much after I also spent the nickel? How much did I spend altogether? How much is 47¢ less 10¢ and 5¢? How much is $47 - (10+5)$?" We can write:

$$47 - 15 = 47 - (10+5) = 47 - 10 - 5$$

You may need many similar examples to develop an understanding of this limitation on regrouping in subtraction.

"John had 88 marbles. How many did he have left after he gave 10 to his twin brother? How many left after he gave 6 more to his younger brother? How many is 88 less 10 and 6? How many did he give away altogether?"

$$88 - 16 = 88 - (10+6) = 88 - 10 - 6 = 72$$

For each example after the first eight, ask members of the class to describe situations for which these sentences might be considered appropriate headlines. You may rewrite examples on the board.

$47 - 15 =$ $\qquad$ $88 - 16 =$

$$\begin{array}{r} 40 + 7 \\ -10 - 5 \\ \hline \end{array} \qquad \begin{array}{r} 80 + 8 \\ -10 - 6 \\ \hline \end{array}$$

Some children may feel uncomfortable by another aspect of some of the examples. We write:

$$40 + 7 - 10 - 5 = 40 - 10 + 7 - 5$$

We are now introducing one additional idea when, as we change the order of the terms, we move + and − signs around; in fact, the operational sign that precedes each term seems to travel with it. But we do not want any such discussion if we can avoid it. Rather, let's use some examples.

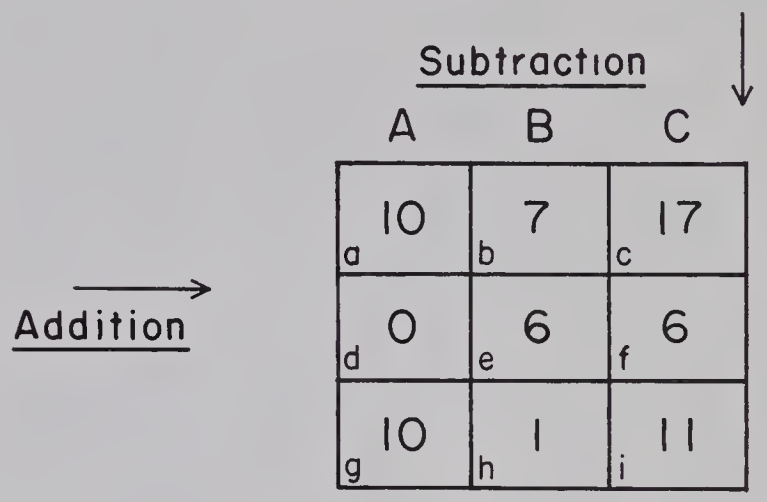
Bill had four dimes and seven pennies—47¢ in all. He spent a dime and then he spent five pennies. How many dimes has he left? How much money is that? (40−10 = 30.) How many pennies did he have and how many did he spend? (7−5 = 2.) So, the expression

$$40 + 7 - 10 - 5 = 40 - 10 + 7 - 5$$

seems to be a most appropriate headline for the happening outlined above.

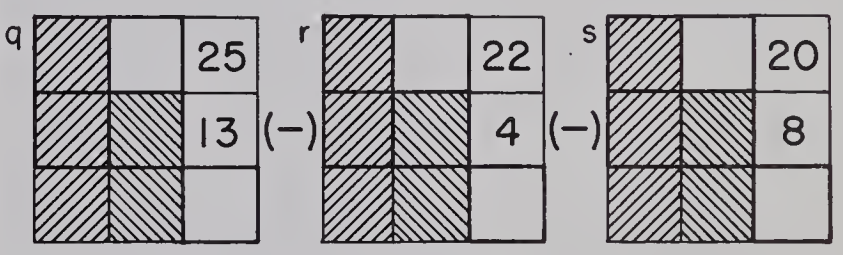
The Cross Number Puzzle and subtraction algorisms

Here is a new use of the Cross Number Puzzle form. It needs some explaining. All horizontal work, from left to right, is addition. All work in columns, top to bottom, is "subtraction." The arrows below indicate the new rules employed:



The rules concerning numbers that are to be written in the first and second columns (A and B) are the same.

Column A is reserved for multiples of 10 (including 0) and column B, for numbers less than 10. These rules require the child to go through each step of the traditional algorisms of subtraction. Here are some further examples:



In preparation for this activity, put some examples involving several levels of difficulty in subtraction on the board.

$12-9 = 3$ because $3 + 9 = 12$

$27-5 =$ because $\_\_ + \_\_ = \_\_$

$36-11 =$ because $\_\_ + \_\_ = \_\_$

$31-19 =$ because $\_\_ + \_\_ = \_\_$

And, using the vertical form

$$\begin{array}{r} 13 \\ -\ 4 \\ \hline 9 \end{array} \quad \text{because} \quad \begin{array}{r} 4 \\ +9 \\ \hline 13 \end{array}$$

Point out that there are always an unlimited number of ways to work your way ahead to a correct solution. Sometimes, you may want to suggest very long methods simply to underline the variety of routes that can be taken.

18 On the next two pages use the following rules:

a. only multiples of 10 (including 0) can be written here.

b. only numbers less than 10 (including 0) can be written here.

c. only numbers that are not multiples of 10 and not less than 10 can be written here.

Complete the following examples:

1. 7 + 3 = 10
2. 7 + 20 = 27
3. 10 + 0 = 10
4. 35 − 30 = 5

Rearrange and regroup the following according to the rules on the last four pages so the result is in the form of + :

1. 17 + 3 → 10 + 7 + 3 → 10 + 10 → 20 + 0
2. 24 + 9 → 20 + 4 + 9 → 20 + 13 → 20 + 10 + 3 → 30 + 3
3. 18 + 16 → 10 + 8 + 10 + 6 → 20 + 14 → 20 + 10 + 4 → 30 + 4
4. 24 + 25 → 20 + 4 + 20 + 5 → 20 + 20 + 4 + 5 → 40 + 9
5. 26 − 7 → 10 + 16 − 7 → 10 + 9
6. 31 − 14 → 20 + 11 − 10 − 4 → 20 − 10 + 11 − 4 → 10 + 7
7. 59 − 26 → 50 + 9 − 20 − 6 → 50 − 20 + 9 − 6 → 30 + 3
8. 37 + 37 → 30 + 7 + 30 + 7 → 60 + 14 → 60 + 10 + 4 → 70 + 4
9. 8 + 54 → 8 + 4 + 50 → 12 + 50 → 2 + 10 + 50 → 60 + 2
10. 93 − 39 → 80 + 13 − 30 − 9 → 80 − 30 + 13 − 9 → 50 + 4

Below are similar examples written in a form used sometimes in poetry:

a. 93 − 39 →
80 + 13
−30 − 9 →
50 + 4

b. 37 + 54 →
30 + 7
+ 50 + 4 →
80 + 11 →
80 + 10 + 1 →
90 + 1

c. 74 + 18 →
70 + 4
+ 10 + 8 →
80 + 12 →
80 + 10 + 2 →
90 + 2

d. 58 − 29 →
40 + 18
− 20 − 9 →
20 + 9

e. 87 + 7 →
80 + 7
0 + 7 →
80 + 14 →
80 + 10 + 4 →
90 + 4

f. 61 − 8 →
50 + 11
− 8 →
50 + 3

g. 90 − 67 →
80 + 10
− 60 − 7 →
20 + 3

h. 99 + 99 →
90 + 9
+ 90 + 9 →
180 + 18 →
180 + 10 + 8 →
190 + 8

19 A Subtraction Game:

1. 5 + 3 − 3 → 5 + 0 → 5
2. 9 − 7 → 2 + 7 − 7 → 2
3. 3 + 5 − 3 → 5 + 3 − 3 → 5
4. 9 − 3 + 7 → 6 + 3 − 3 + 7 → 6 + 7 → 13
5. 8 − 3 − 4 → 5 + 3 − 3 − 4 → 5 − 4 → 1 + 4 − 4 → 1
6. 9 − 1 − 6 → 8 + 1 − 1 − 6 → 8 − 6 → 2 + 6 − 6 → 2
7. 15 − 7 − 8 → 8 + 7 − 7 − 8 → 8 − 8 → 0
8. 13 − 4 − 5 → 9 + 4 − 4 − 5 → 9 − 5 → 4 + 5 − 5 → 4

The following examples require some rearranging:

9. 3 − 5 + 6 → 3 + 6 − 5 → 3 + 1 + 5 − 5 → 3 + 1 → 4
10. 2 − 4 + 3 → 2 + 3 − 4 → 5 − 4 → 1 + 4 − 4 → 1
11. 0 − 5 + 6 → 0 + 6 − 5 → 6 − 5 → 1 + 5 − 5 → 1
12. 7 − 8 + 1 → 7 + 1 − 8 → 8 − 8 → 0
13. 3 − 5 + 7 → 3 + 7 − 5 → 3 + 2 + 5 − 5 → 3 + 2 → 5
14. 8 − 3 − 1 → 5 + 3 − 3 − 1 → 5 − 1 → 4 + 1 − 1 → 4
15. 3 − 8 + 6 → 3 + 6 − 8 → 9 − 8 → 1 + 8 − 8 → 1

Mark the following statements as True or False:

	T	F		T	F
a 3 + 7 = 1 + 3 + 7 − 1	✓		A 24 + 39 = 23 + 40	✓	
b 27 = 4 + 27 − 4	✓		B 85 − 7 = 88 − 10	✓	
c 8 − 6 = 8 − 1 + 6 + 1		✓	C 25 + 7 = 28 + 10		✓
d 4 − 1 = 1 + 4 − 1 + 1		✓	D 71 − 71 = 70 − 70	✓	
e 7 + 10 = 7 − 6 + 10 + 6	✓		E 21 + 21 = 20 + 20		✓
f 13 − 3 = 8 + 13 − 8 − 3	✓		F 93 − 87 = 90 − 84	✓	
g 19 + 38 = 19 + 1 + 38 − 1	✓		G 93 − 87 = 96 − 90	✓	
h 38 − 19 = 38 + 1 − 19 + 1		✓	H 97 − 68 = 86 − 79		✓
i 68 + 25 = 2 + 68 + 25 − 2	✓		I 80 − 39 = 81 − 40	✓	
j 37 + 37 = 37 − 3 + 37 + 3	✓		J 145 − 39 = 146 − 40	✓	
k 34 + 56 = 23 + 4 − 25 + 6		✓	K 262 + 98 = 260 + 100	✓	
l 25 + 35 = 25 + 5 + 35 − 5	✓		L 321 − 182 = 339 − 200	✓	
m 29 + 13 = 13 + 6 + 29 − 6	✓		M 268 + 286 = 270 + 280		✓
n 87 − 68 = 3 + 87 − 68 + 3		✓	N 1003 − 995 = 1008 − 1000	✓	
o 44 − 35 = 44 − 5 − 35 + 5	✓		O 555 + 950 = 605 + 900	✓	

$$
\begin{aligned}
30-16 &= (30-4)-4-4-4\\
&= (26-4)-4-4\\
&= (22-4)-4\\
&= 18-4\\
&= 14
\end{aligned}
$$

$$
\begin{aligned}
30-16 &= (30-10)-10+4\\
&= (20-10)+4\\
&= 10+4
\end{aligned}
$$

Children need to understand that they have already acquired many tools that can be used to reduce large problems down to manageable size. Give two or three examples:

$92 - 27 =$ ____

$$\begin{array}{r} 115 \\ -\ 28 \\ \hline \end{array}$$

"Now, get out your tools and go to work." If you are patient, these—and even more difficult—examples can be solved without resort to the algorisms that are eventually learned as shortcuts.

$$\begin{array}{r} 18 \\ -\ 13 \\ \hline 5 \end{array} \quad \text{because} \quad \begin{array}{r} 5 \\ +\ 13 \\ \hline 18 \end{array}$$

$$\begin{array}{r} 23 \\ -\ 17 \\ \hline \end{array} \quad \text{because} \quad +\_\_\_\_$$

$$\begin{array}{r} 52 \\ -\ 28 \\ \hline \end{array} \quad \text{because} \quad +\_\_\_\_$$

The order in which the addends are written is entirely immaterial: 12 − 9 = 3 because 3 + 12, or because 9 + 3 = 12. It is often well to write the addends in both orders to emphasize this point. Sometimes, use a shorthand form: 12 − 9 = 3, because 3 + 9 = 12.

After the examples involving subtraction are written on the board, ask volunteers to tell how each could be completed. "There are many ways to get to a result. How did you get there?"

Some may count; others will go through quite involved processes; some may even use techniques that work in the example at hand by coincidence, but would lead to wrong conclusions in another example. Give each technique its day in court.

Any number subtracted from the same number

The top section of *Work Text* page 19 focuses upon the idea that any number subtracted from itself equals zero. By regrouping some numbers and, from example 9 on, by rearranging some numbers first, a result is achieved by eliminating all but one number. Thus,

in example 5: $8-3-4 \rightarrow \underline{5+3}-3-4 \rightarrow \underline{5}-4 \rightarrow \underline{1+4}-4 \rightarrow \underline{1}$
in example 9: $3-5+6 \rightarrow 3+\underline{6}-5 \rightarrow 3+\underline{1+5}-5 \rightarrow \underline{3}+1 \rightarrow \underline{4}$

The bottom section is an excellent test of the extent to which the ideas of regrouping and the In-Any-Order Rule have been mastered. Pupils should tackle this section with no preparation by the teacher.

Extended work in regrouping

"Here is a group of additon examples that look very complicated. How long do you think it would take you to work out all the answers?"

$100 + 20 + 3 =$ ____
$300 + 70 + 7 =$ ____
$200 + 50 + 6 =$ ____
$600 + 10 + 3 =$ ____
$700 + 40 + 4 =$ ____

Here is another group:

$3 + 5 + 6 + 3 + 4 =$ ____
$20 + 70 + 50 + 10 + 40 =$ ____
$100 + 300 + 200 + 600 + 700 =$ ____

"Take your time. These problems are more difficult." After there is general agreement on the results, fill them in.

"In the first group of additon problems, the numbers are written in three columns. Can anyone tell me the sum of the numbers written in the first column on the left? The sum of the numbers written in the second column? The sum of the numbers written in the third column?

"In the second group, can you tell me the sum of the numbers written in the left hand column? And in each other column?" Easy, again.

"Now, here are some problems that look most difficult."

1900	123
190	375
+ 21	256
	613
	+ 744

"Did you expect that result? Why? What is the sum of the numbers in the answer column of the first series? What is the sum of the numbers written in the answer column of the second series?

"Here is a group of problems that are something like the first group, but a little more difficult. Let's see how fast we can go through them." You may want to divide the class into teams that compete—a member of each taking alternate examples. Or, establish an order and go around the room.

A.	B.
$100 + 20 + 4 =$ ____	$200 + 120 + 8 =$ ____
$200 + 50 + 14 =$ ____	$500 + 250 + 2 =$ ____
$100 + 70 + 25 =$ ____	$400 + 410 + 9 =$ ____
$400 + 70 + 30 =$ ____	$300 + 95 + 5 =$ ____
$400 + 70 + 37 =$ ____	$200 + 180 + 20 =$ ____

C.	D.
$200 + 120 + 15 =$ ____	$150 + 70 + 6 =$ ____
$500 + 180 + 11 =$ ____	$200 + 80 + 1 =$ ____
$300 + 190 + 11 =$ ____	$180 + 30 + 5 =$ ____
$700 + 150 + 55 =$ ____	$270 + 40 + 12 =$ ____
$800 + 75 + 80 =$ ____	$580 + 430 + 25 =$ ____

There is a certain scaling in this series and within each series. Watch carefully for the examples that cause more hesitation than others. Build other series of examples that expand in the areas in which lack of sureness is displayed.

Notes to accompany *Work Text* pages 20-31

Pages 20 through 31 have a story line based on a new boy at Roosevelt School in Central City, Ben Harris. Mrs. Barnes, the teacher, wants to find out how far Ben has come along in developing his ability to compute. She finds he has no trouble at all with addition, but seems quite limited in his ability to multiply. He knows only those multiplication facts in which neither factor is greater than 5.

But he accomplishes surprising results with this limited knowledge.

The purpose behind the story

On the surface, the story form used for these pages may appear as simply a means to include exercise in recalling the easier multiplication facts and in performing additions involving more than two addends. The real purpose, however, is much more fundamental.

The devices that Ben employs to make up for his limited knowledge of multiplication facts are precisely the same devices we all use to make up for our limited knowledge of multiplication facts. We generally employ only the facts in which neither factor is larger than 9, a multiple of 10 that is not larger than 9×10, or a multiple of 100 that is not larger than 9×100, etc.

"But I have a shorter way to find the answer to 14 × 3. I do it this way."

"You see! I Do-It-To-All-Three. Sometimes," Ben added, "I have to Do-It-To-All-Four. Here's an example:"

"How about 14 3?", Sue asked. Ben wrote without saying anything.

Mrs. Barnes was impressed. So were others in the class. Some were puzzled.

"Will your Do-It-To-Both always work?"

"It always has," Ben answered.

Mrs. Barnes suggested, "OK, we multiply Ben's way for a while."

"Remember," she added, "you must never use any multiplication or division fact in which any of the factors is more than five."

"I think Do-It-To-Both is a silly idea to use in multiplying and dividing," Anne said as she looked at Ben.

"It works," was Ben's only reply.

"Well, Anne, after we've learned more about Ben's way, we will let you show Ben your way of multiplying and dividing."

DO-IT-TO-BOTH

2 x 10 = 2 x (5 + 5) = 10 + 10 = 20 20 ÷ 2 = (10 + 10) ÷ 2 = 5 + 5 = 10

10 x 3 = (5 + 5) x 3 = 15 + 15 = 30 18 ÷ 2 = (10 + 8) ÷ 2 = 5 + 4 = 9

2 x 9 = 2 x (4 + 5) = 8 + 10 = 18 30 ÷ 3 = (15 + 15) ÷ 3 = 5 + 5 = 10

8 x 3 = (5 + 3) x 3 = 15 + 9 = 24 24 ÷ 3 = (12 + 12) ÷ 3 = 4 + 4 = 8

DO-IT-TO-ALL-THREE

2 x 14 = 2 x (5 + 5 + 4) = 10 + 10 + 8 = 28

15 x 3 = (5 + 5 + 5) x 3 = 15 + 15 + 15 = 45

30 ÷ 2 = (10 + 10 + 10) ÷ 2 = 5 + 5 + 5 = 15

60 ÷ 5 = (20 + 20 + 20) ÷ 5 = 4 + 4 + 4 = 12

5 x 13 = 5 x (5 + 5 + 3) = 25 + 25 + 15 = 65

48 ÷ 4 = (16 + 16 + 16) ÷ 4 = 4 + 4 + 4 = 12

70 ÷ 5 = (25 + 25 + 20) ÷ 5 = 5 + 5 + 4 = 14

The next day, Mrs. Barnes asked Anne what she thought of Ben's way of multiplying and dividing. "You mean that Do-It-To-Both stuff? Why does it work only when one of the factors is 5 or less, such as 4 × 14. "Can I give Ben a simple problem that he can't do with his Do-It-To-Both unless he knows much more than 5 × 5 = 25?" "Surely, Anne. Put your problem on the board."

Anne wrote:

Betty Grant raised her hand. "I think I can do that one Ben's way. May I try?" Mrs. Barnes said, "Of course." Here is Betty's work:

"Is that right, Ben?" Betty asked. "Yes. However, I would have done it with a few less steps." "Will you show us, Ben?"

Here is Ben's work:

ERROR:

8 × 4 = 4 × 4 = 16 + 16 + 16 + 16 = 64

"I call that the Double-Do-It-To-Both method." "Why did you pick two factors that are the same number?" said Anne. "How about 7 × 9?" Ben didn't hesitate.

7 × 9 = 15 + 13 + 13 + 13 = 25 + 20 + 8 + 63

Anne's face fell. "Well, that's a kind of multiplication that's sure full of addition.

"I learned addition before I knew about multiplication, so I use that instead of remembering so many multiplication facts."

"Will it always work, Ben?" Betty asked.

"I've never known it to fail," was Ben's quiet answer

"Can we try some examples, Mrs. Barnes?" Jerry wanted to know. "Yes, I'll put some on the board."

Anne looked at Ben. "Does he really know what this is all about?" she wondered.

"Remember," Mrs. Barnes said, "You must imagine that you don't know how to multiply if any factor is greater than 5.

Much of the insight into this parallelism will be lost if it is discussed with the children before they *discover* this relationship. And the eventual discovery will be invaluable in terms of understanding the algorism of multiplication.

Ben argues that 4 × 7 can be approached as follows:

$$4 \times 7 = 4 \times (5+2) = 4 \times 5 + 4 \times 2 = 20 + 8 = 28$$

When we are confronted with finding the product when 4 and 16 are the factors, we argue that it can be approached as follows:

$$4 \times 16 = 4 \times (10+6) = 4 \times 10 + 4 \times 6 = 40 + 24 = 64$$

But, too often, we see this in a form that soon obscures this argument. We get to a result in the following fashion:

$$\begin{array}{r} 16 \\ \times\ 4 \\ \hline 64 \end{array}$$

And most of us say to ourselves: "4 times 6 is 24, put down the 4 and carry the 2 (tens); 4 times 1 (ten) is 4 (tens); plus the 2 (tens) I carried, is 6, put down the 6 . . . the product is 64." (And we probably no longer say those parenthetical "tens" to ourselves.)

What we are really doing is a little more explicit if we write down our thinking this way:

$$\begin{array}{r} 16 \\ \times\ 4 \\ \hline 64 \end{array} \rightarrow \begin{array}{r} 10 + 6 \\ \times\ 4 \\ \hline 24 \\ +\ 40 \\ \hline 64 \end{array}$$

In the story narrated on these 12 pages, we learn that Ben's sister makes a record of the "Do-It-to-Both" device in a setting that is more familiar. She finds that 4 × 7 = 28 in this way:

$$\begin{array}{r} 5 + 2 \\ \times\ 4 \\ \hline 8 \\ +\ 20 \\ \hline 28 \end{array}$$

In these examples, a general principle of arithmetic is involved. Ben calls the principle his Do-It-to-Both Rule; most mathematicians refer to it as the *distributive principle* (short for the *principle that multiplication is distributive over addition).* The teacher should use his own discretion in the terminology he uses with the children. The authors are content, at this level of development, to let Do-It-to-Both point to the pattern involved.

Some may want to give several examples of the pattern, and then use letters to generalize the idea, such as the following:

$$3\times12 = 3\times(7+5) = 3\times7 + 3\times5$$
$$8\times12 = 7\times(10+12) = 8\times10 + 8\times2$$
$$6\times45 = 6\times(40+5) = 6\times40 + 6\times5$$

$$a\times(b+c) = a\times b + a\times c$$

The Do-It-to-Both formulation has a distinct advantage in lending itself to expansion of the idea into the Do-It-to-All-Three Rule (page 21) and the Double-Do-It-to-Both Rule (pages 22 and 23) and the Double-Do-It-to-All-Three Rule (page 23), etc.

(If the *distributive principle* of the mathematicians is used, one would have to be very cautious in enlarging upon the idea without an involved bit of mathematical reasoning for which the children are not prepared at this point.)

The teacher will be better prepared for handling Ben's story if he at once familiarizes himself with page 24, where Mrs. Barnes uses diagrams or sketches to help explain why Ben's several rules work, and lead to reasonable conclusions.

At this point children ought to be asked to find examples of Ben's rules as they apply it in the world of things. A use of the *headline* technique outlined in the *Problem Solving* section of the Preface of this guide is in order. Here is a sample headline:

$$5 \times 7 = 5 \times (4+3) = 20 + 15 = 35$$

Appropriate stories might be: "Each of five space capsules was manned by a crew of seven—four helpers and three pilots. There would be 20 helpers in all for 15 pilots, or a total of 35 crewmen."

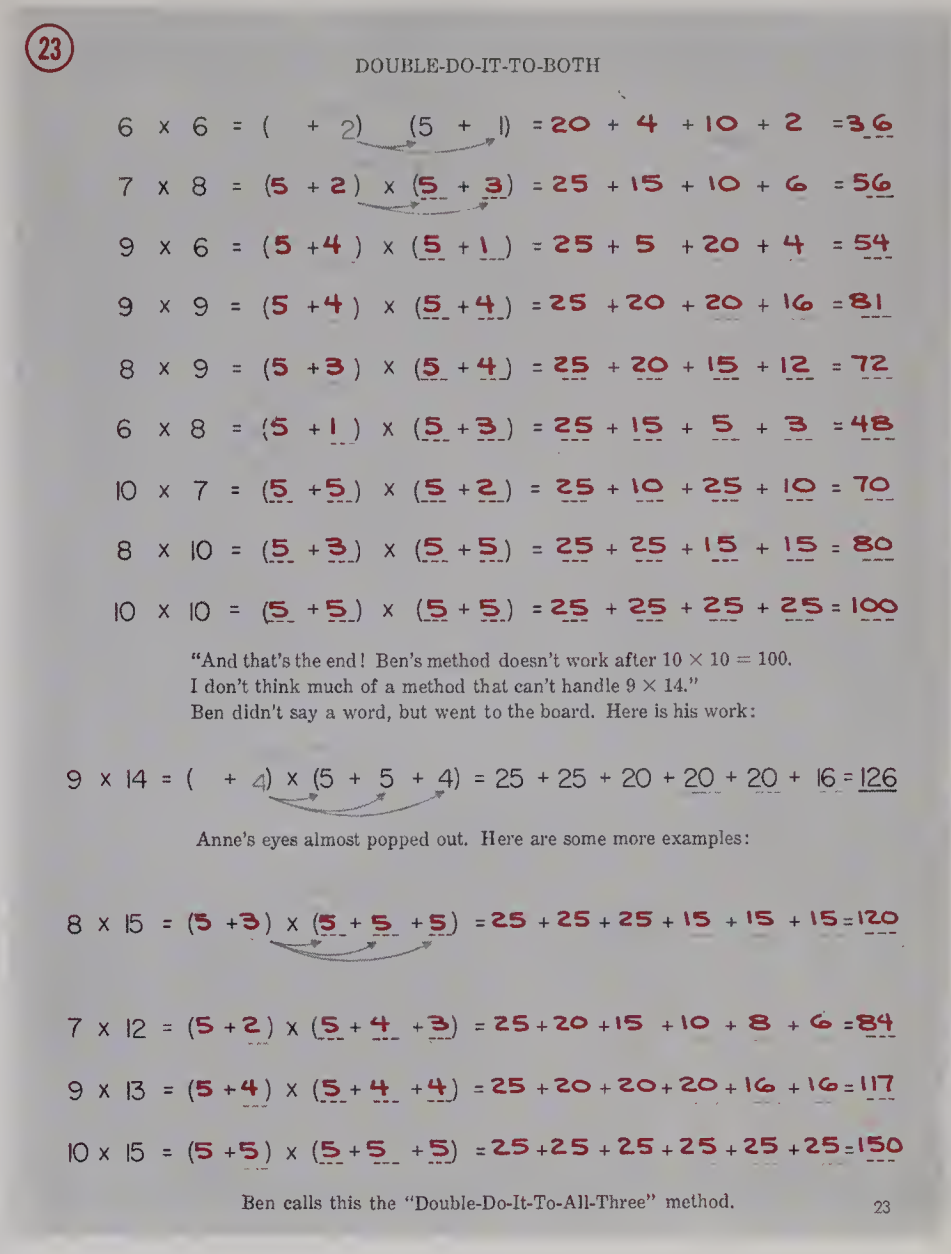
"In each of five rows of seven children, there were four girls and three boys. That would be a total of 20 girls and 15 boys—or, 35 children in all."

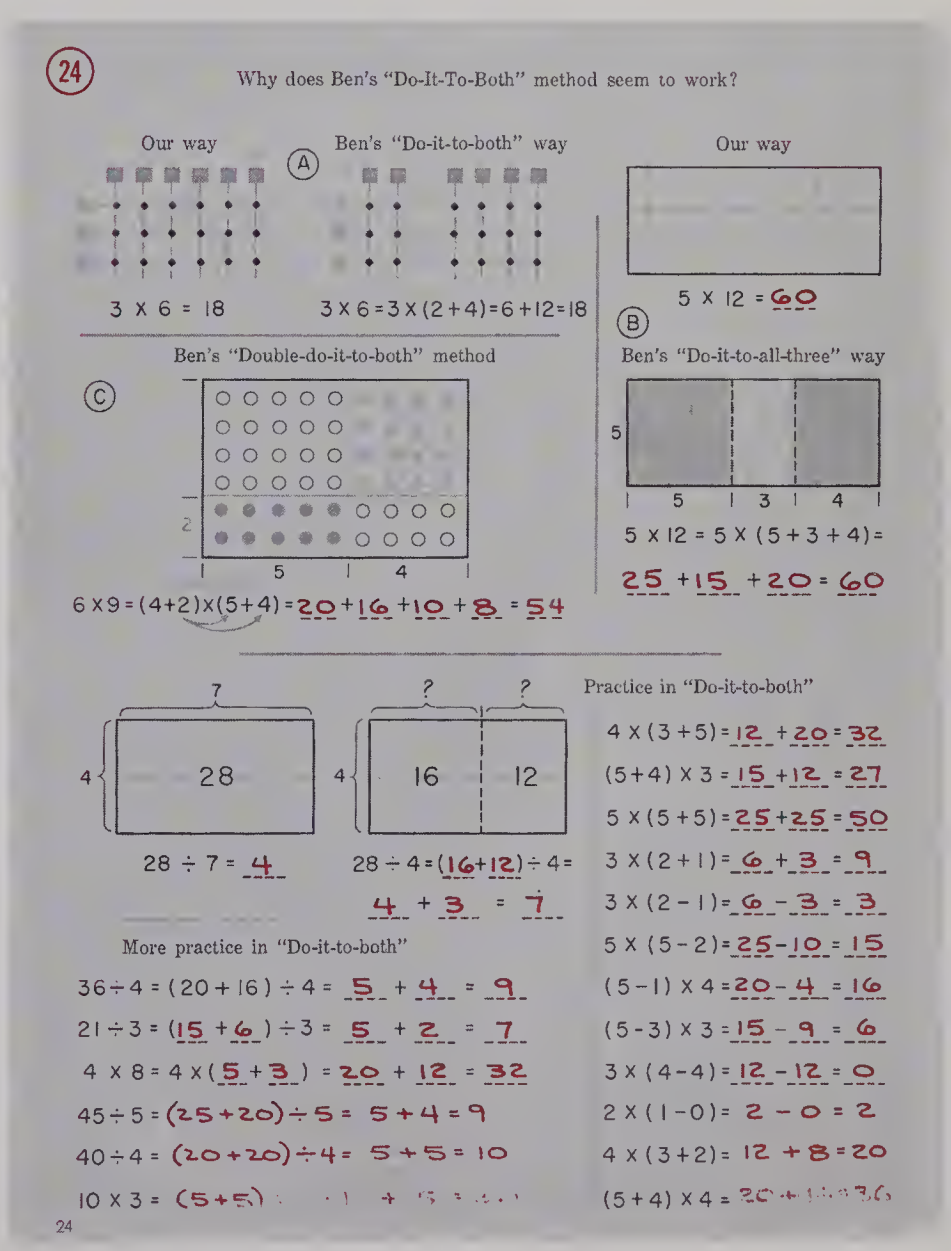
"During a five-week period, there were only four days of school each week and 3 holidays. There were five more days of school than holidays because there were 20 days of school and 15 holidays out of a total of 35 days."

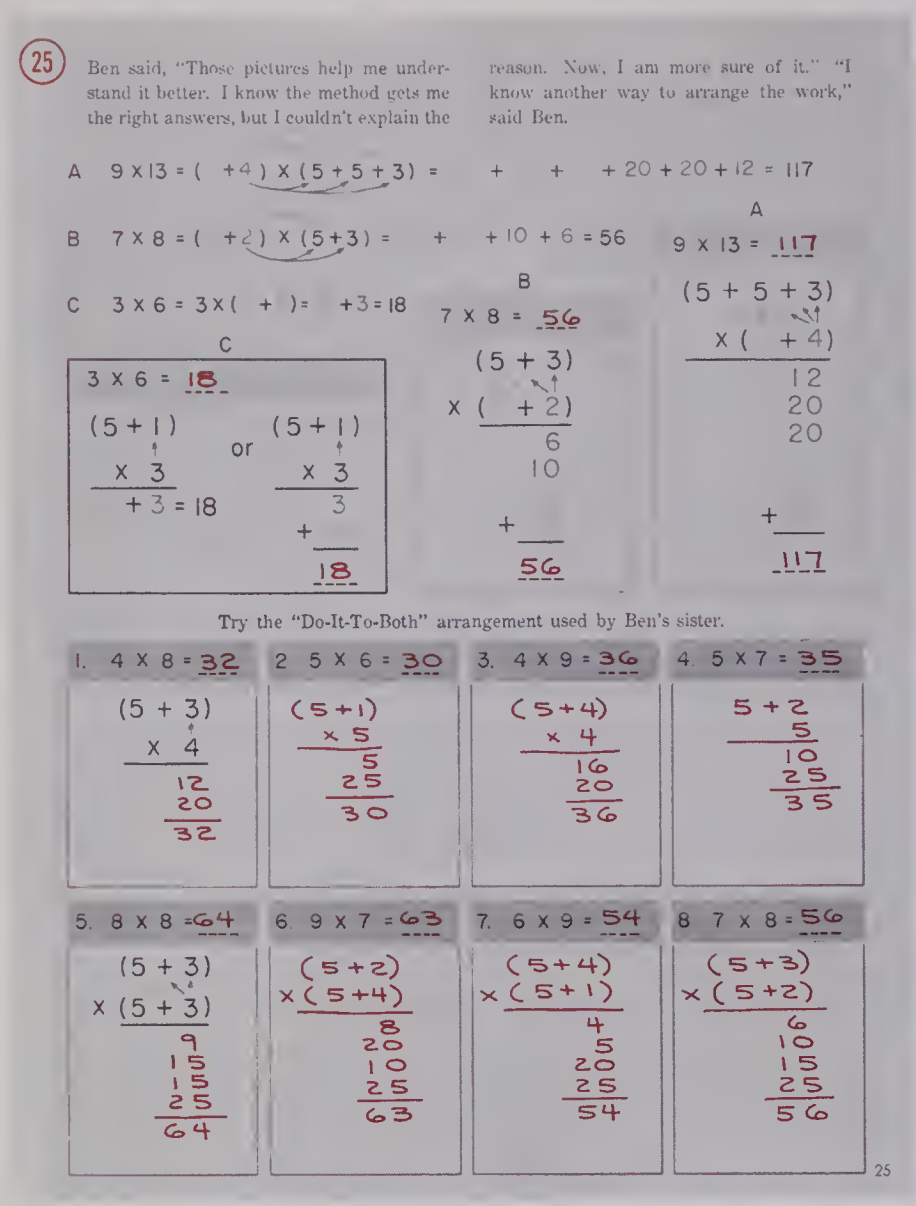
Another headline might be:

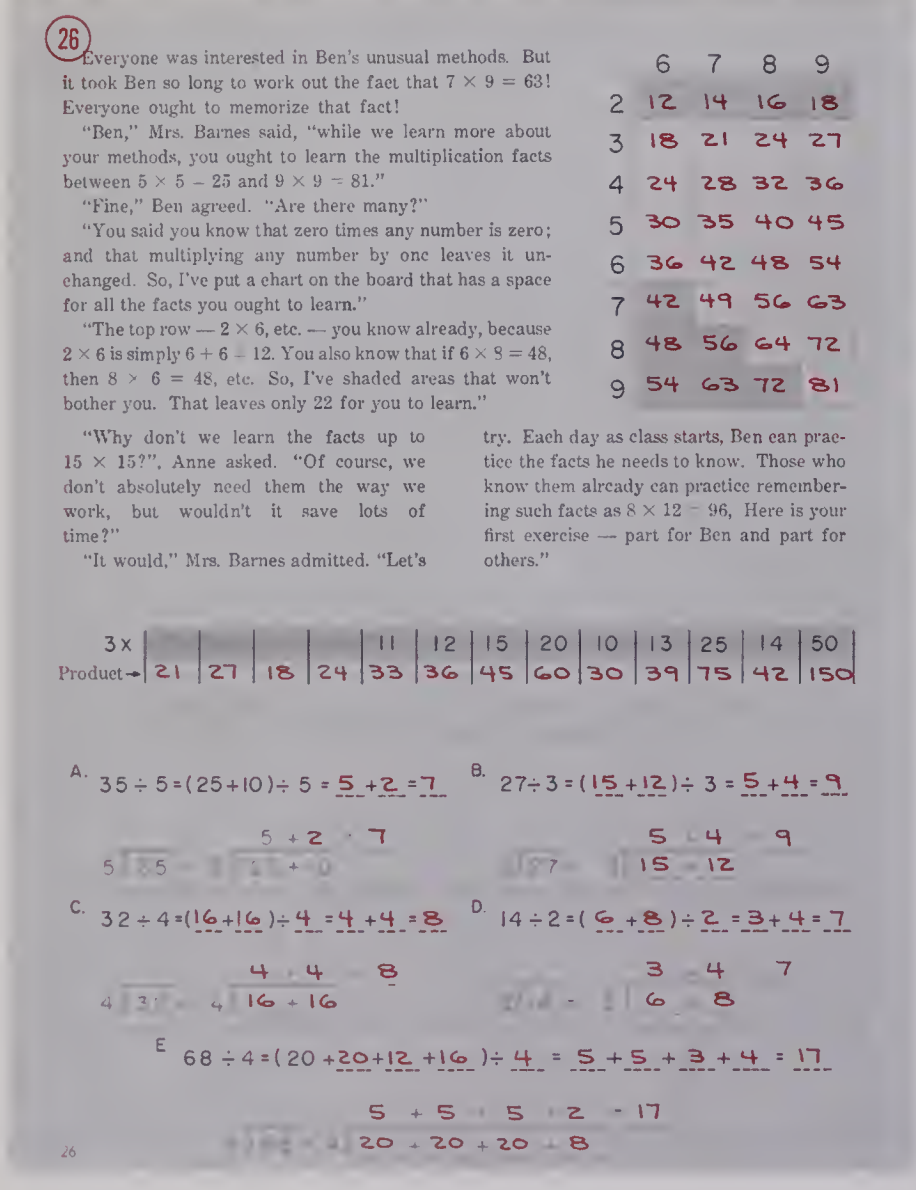
$$3\ (10+4) = 3 \times 10 + 3 \times 4 = 30 + 12 = 42$$

"Each of three children saved one dime and four pennies. Together they had three dimes and twelve pennies—or 42¢ in all."






25
Ben said, "Those pictures help me understand it better. I know the method gets me the right answers, but I couldn't explain the reason. Now, I am more sure of it." "I know another way to arrange the work," said Ben.
A 9 x 13 = (+4) x (5 + 5 + 3) = + + + 20 + 20 + 12 = 117
B 7 x 8 = (+2) x (5+3) = + +10 + 6 = 56
C 3 x 6 = 3x(+)= +3=18
C
3 x 6 = 18
(5 + 1) x 3 + 3 = 18
or
(5 + 1) x 3 3 + 18
B
7 x 8 = 56
(5 + 3) x (+ 2) 6 10 + 56
A
9 x 13 = 117
(5 + 5 + 3) x (+ 4) 12 20 20 + 117
Try the "Do-It-To-Both" arrangement used by Ben's sister.
1. 4 x 8 = 32
(5 + 3) x 4 12 20 32
2 5 x 6 = 30
(5+1) x 5 5 25 30
3. 4 x 9 = 36
(5+4) x 4 16 20 36
4. 5 x 7 = 35
5 + 2 5 10 25 35
5. 8 x 8 = 64
(5 + 3) x (5 + 3) 9 15 15 25 64
6. 9 x 7 = 63
(5+2) x (5+4) 8 20 10 25 63
7. 6 x 9 = 54
(5+4) x (5+1) 4 5 20 25 54
8 7 x 8 = 56
(5+3) x (5+2) 6 10 15 25 56
25


26
Everyone was interested in Ben's unusual methods. But it took Ben so long to work out the fact that 7 × 9 = 63! Everyone ought to memorize that fact!
"Ben," Mrs. Barnes said, "while we learn more about your methods, you ought to learn the multiplication facts between 5 × 5 = 25 and 9 × 9 = 81."
"Fine," Ben agreed. "Are there many?"
"You said you know that zero times any number is zero; and that multiplying any number by one leaves it unchanged. So, I've put a chart on the board that has a space for all the facts you ought to learn."
"The top row — 2 × 6, etc. — you know already, because 2 × 6 is simply 6 + 6 = 12. You also know that if 6 × 8 = 48, then 8 × 6 = 48, etc. So, I've shaded areas that won't bother you. That leaves only 22 for you to learn."
6 7 8 9
2 12 14 16 18
3 18 21 24 27
4 24 28 32 36
5 30 35 40 45
6 36 42 48 54
7 42 49 56 63
8 48 56 64 72
9 54 63 72 81
"Why don't we learn the facts up to 15 × 15?", Anne asked. "Of course, we don't absolutely need them the way we work, but wouldn't it save lots of time?"
"It would," Mrs. Barnes admitted. "Let's try. Each day as class starts, Ben can practice the facts he needs to know. Those who know them already can practice remembering such facts as 8 × 12 = 96. Here is your first exercise — part for Ben and part for others."
3x | 11 | 12 | 15 | 20 | 10 | 13 | 25 | 14 | 50
Product→ | 21 | 27 | 18 | 24 | 33 | 36 | 45 | 60 | 30 | 39 | 75 | 42 | 150
A. 35 ÷ 5 = (25+10) ÷ 5 = 5 + 2 = 7
5 + 2 = 7
B. 27 ÷ 3 = (15 + 12) ÷ 3 = 5 + 4 = 9
5 + 4 = 9
C. 32 ÷ 4 = (16+16) ÷ 4 = 4 + 4 = 8
4 + 4 = 8
D. 14 ÷ 2 = (6 + 8) ÷ 2 = 3 + 4 = 7
3 + 4 = 7
E 68 ÷ 4 = (20 + 20 + 12 + 16) ÷ 4 = 5 + 5 + 3 + 4 = 17
5 + 5 + 3 + 2 = 17
20 + 20 + 12 + 8
26

"Three children divided 42¢ equally by each taking one dime and four pennies."

"An empty box weighed four pounds. Ten pounds of sand was packed in each of three such boxes. The three empty boxes weighed 12 pounds and they were packed with a total of 30 pounds of sand. The loaded boxes weighed 42 pounds."

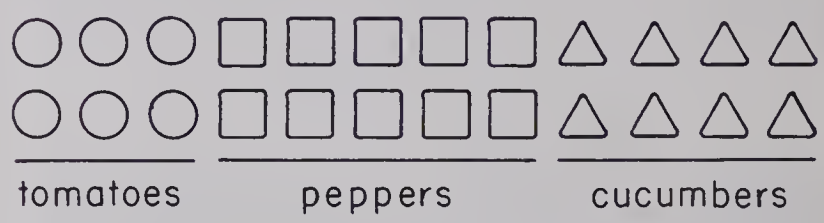
"Jenny had 42 cookies to put up in three packages. She had 30 star-shaped cookies and 12 round ones; so, she put ten star-shaped cookies in each package along with three round ones."

The idea of Do-It-to-Both is so vitally important to understanding the algorisms of multiplication and division that the teacher can afford much time in relating each mathematical expression of each variation of the rule to the world of real events.

Here is another headline:

$$2\,(3+5+4) = 2 \times 3 + 2 \times 5 + 2 \times 4$$
$$= 6 + 10 + 8 = 24$$

"In each of two rows, my father put out three tomato plants, five pepper plants and four cucumber plants. That's six tomato, ten pepper and eight cucumber plants, or two dozen in all." Sketches illustrating the stories will help:



"My brother and I worked during the summer vacation for our uncle. One week we each worked three hours; another week for five hours; and a third week we each worked four hours. That's a total of 24 hours—six hours the first week, ten hours the second week, and ten hours the third week."

Another headline:

$$(2+3) \times (2+4) = 2\times2 + 2\times4 + 3\times2 + 3\times4$$
$$= 4 + 8 + 6 + 12 = 30$$

"In each row there were two girls and three boys. The teacher asked the students in the first two rows to find the product of 11 and 15. She asked the back four rows to find the product of 12 and 14. How many children were there in the class?" Now another sketch is in order:

G G B B B
G G B B B $11 \times 15 =$ ____

G G B B B
G G B B B
G G B B B
G G B B B $12 \times 14 =$ ____

"How many boys found 165 as the product of the factors given them? How many girls found the same product?"

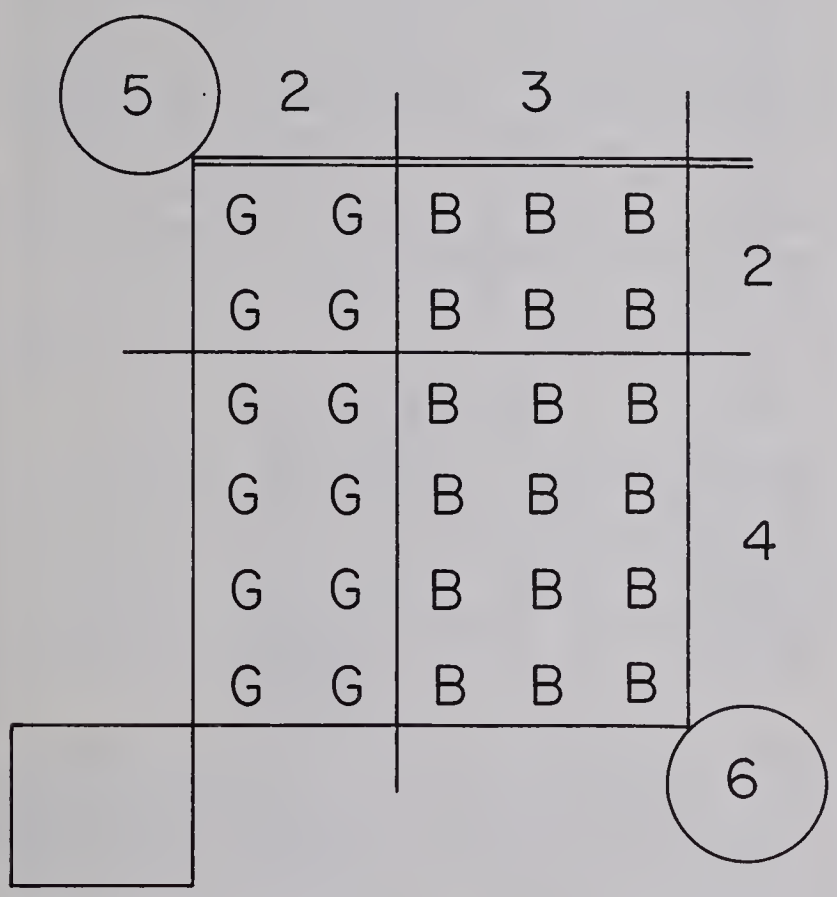
"How many boys in the class? How many girls?"

"How many students in all?"

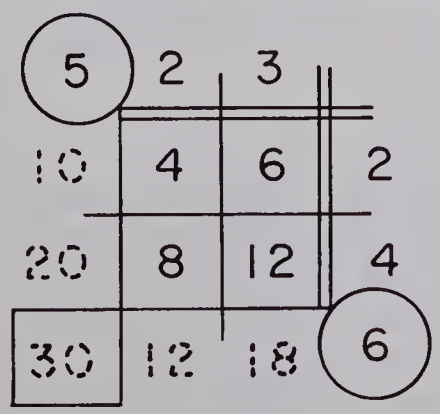
"How many found 168 as the product of the factors given?"

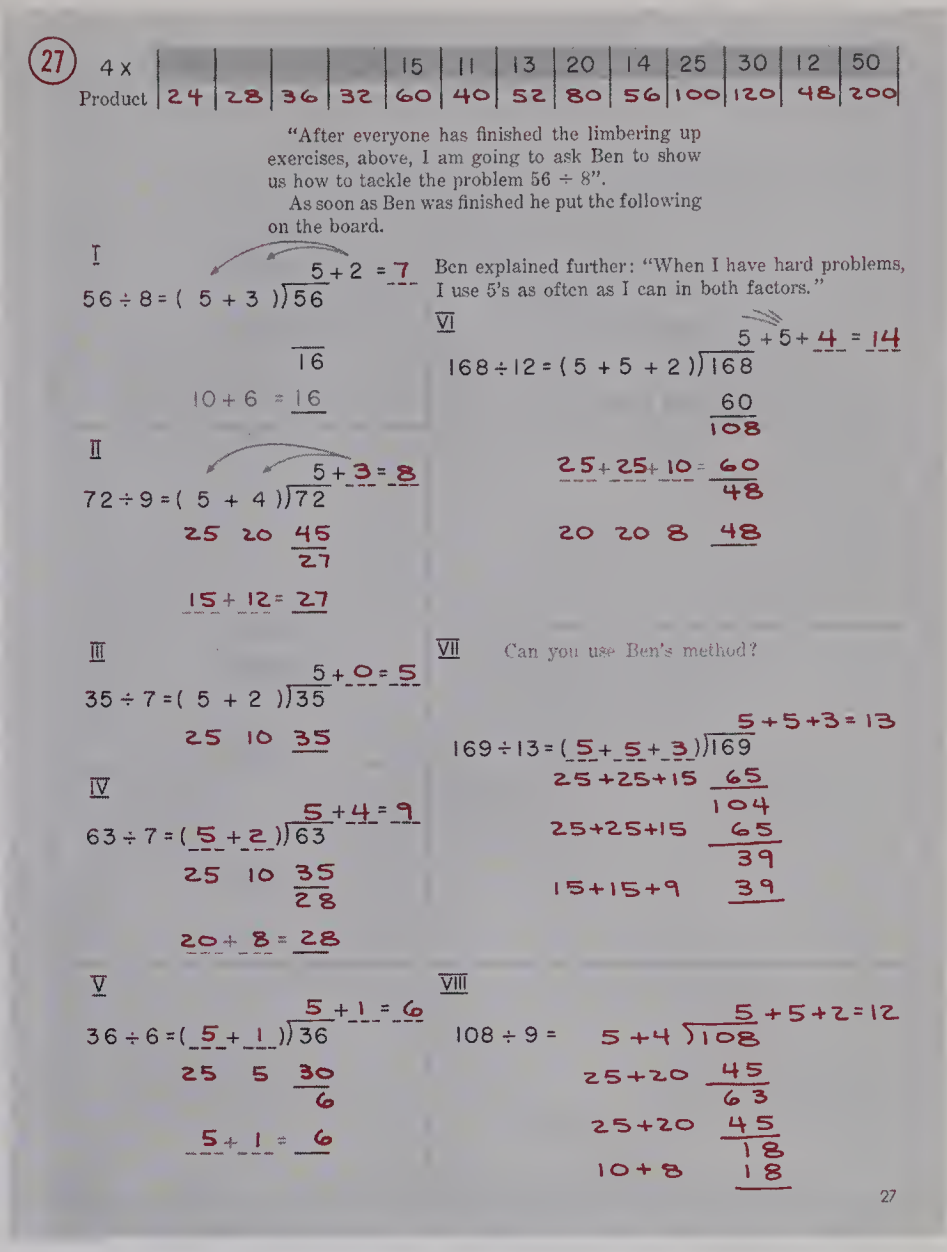
"How many more boys than girls?"

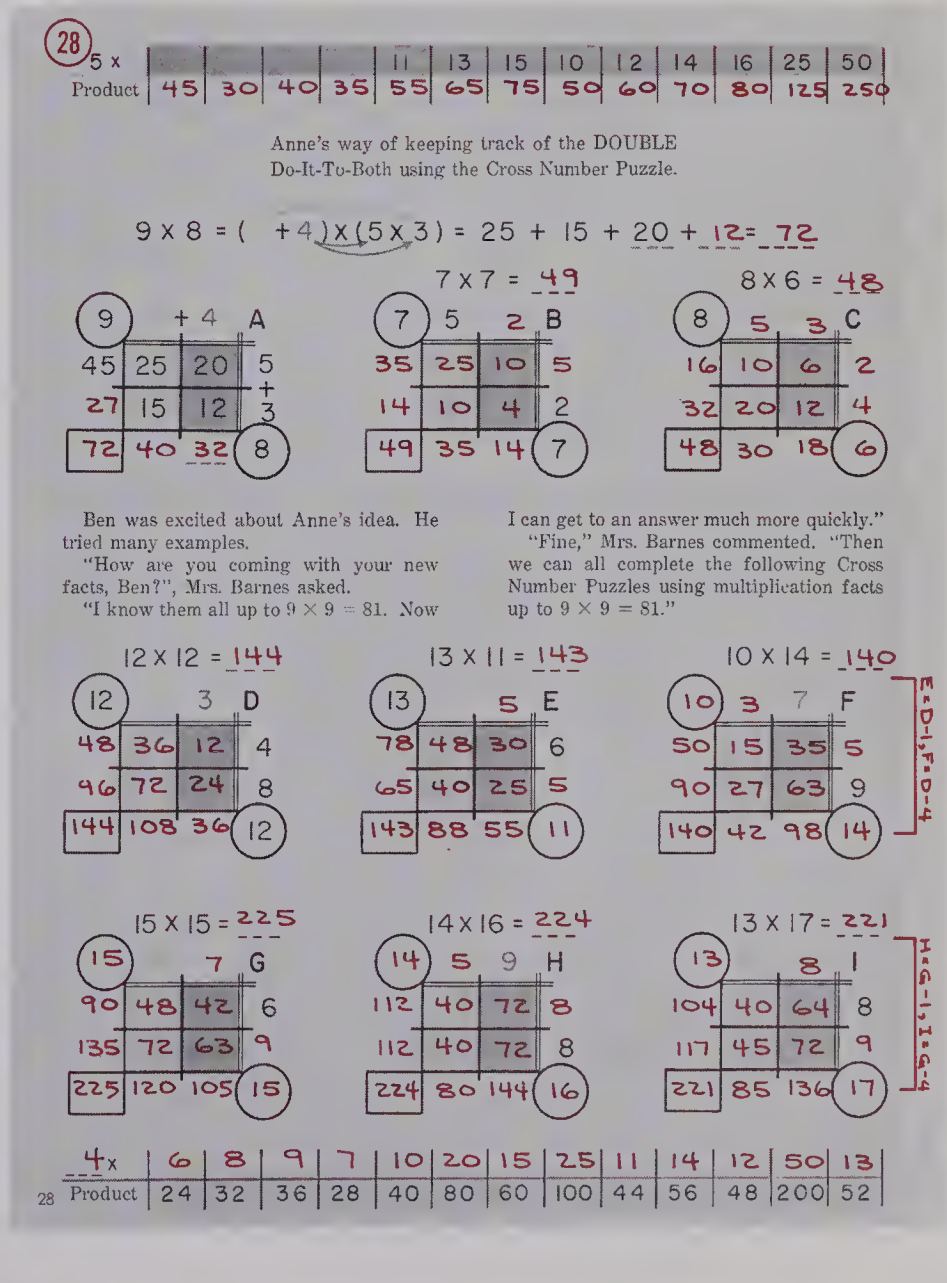
The example explores the use of a Cross Number Puzzle form used on pages 28 through 31:

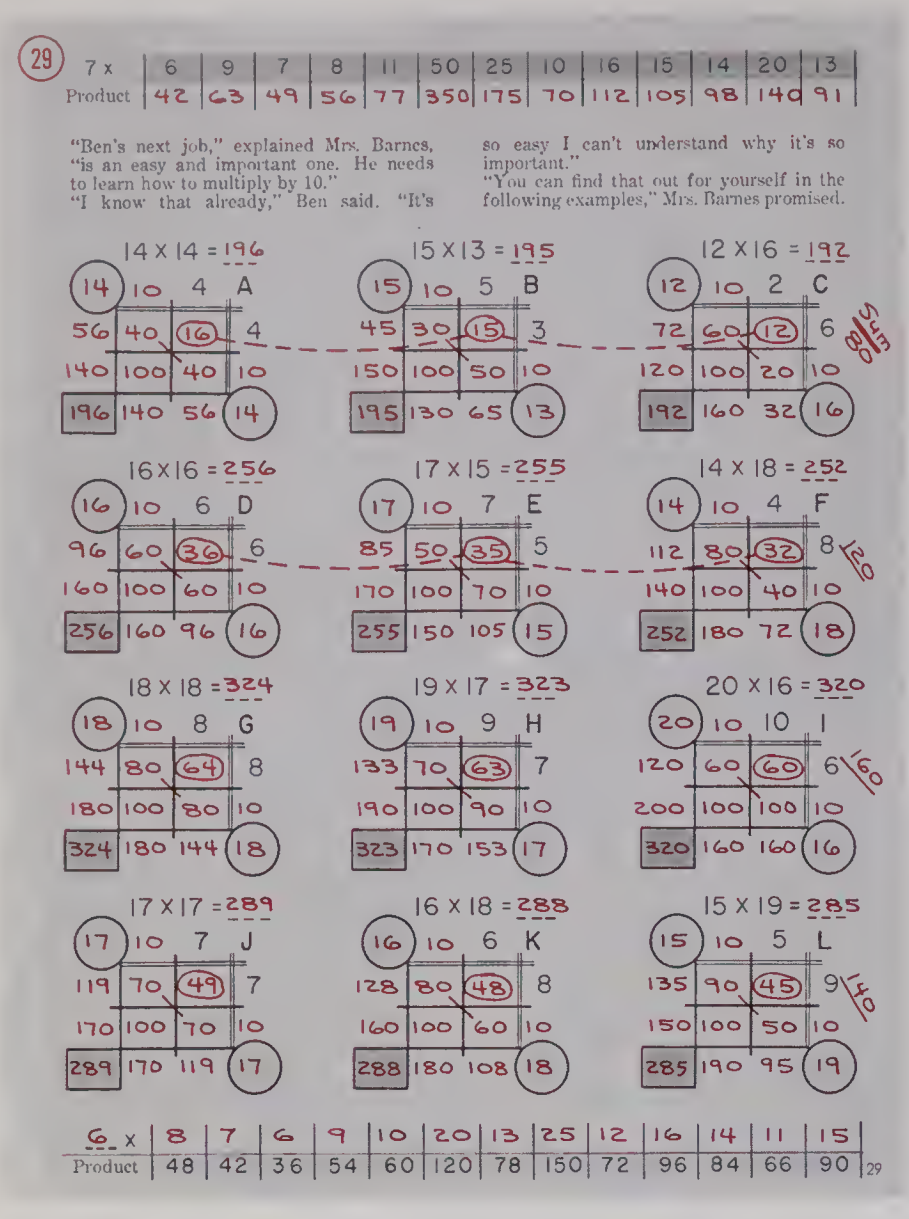


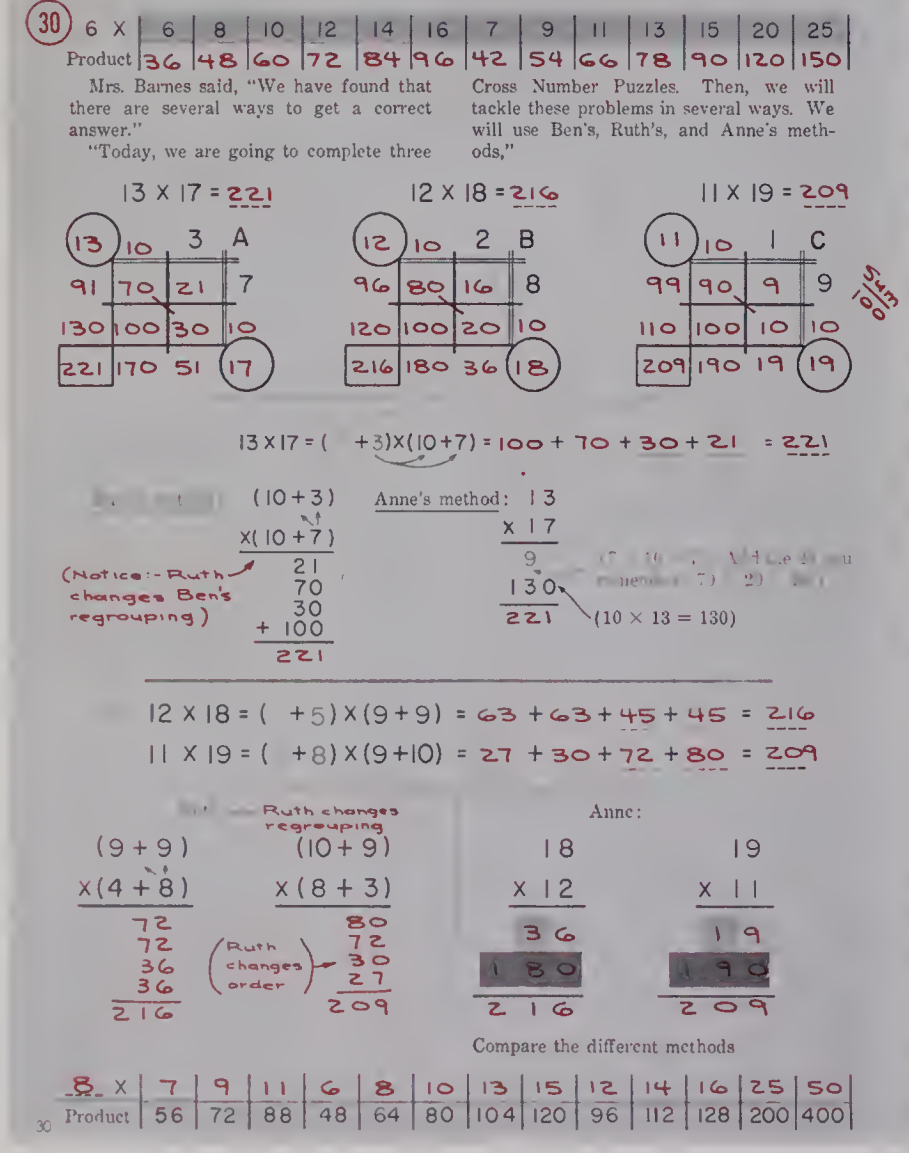
A shorter record looks like the following:







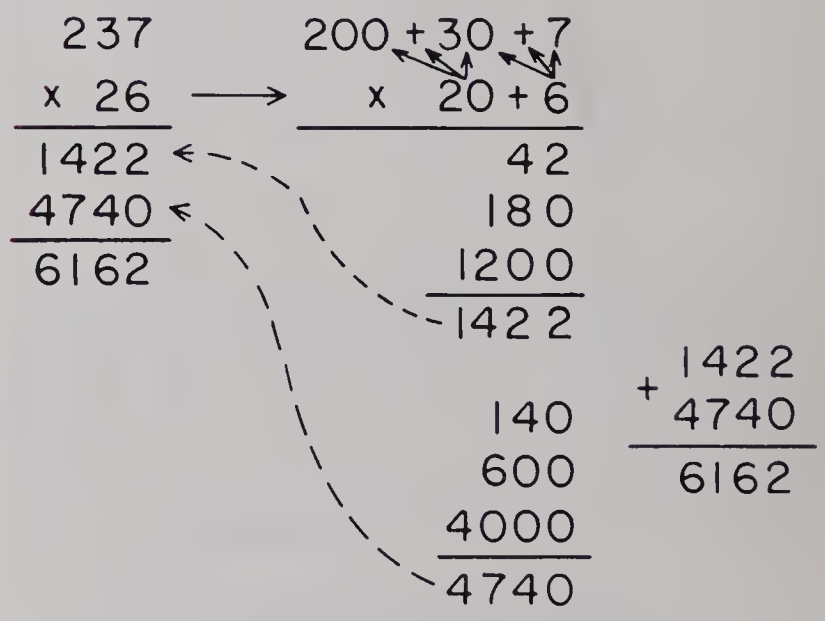

29
7 x | 6 | 9 | 7 | 8 | 11 | 50 | 25 | 10 | 16 | 15 | 14 | 20 | 13
Product | 42 | 63 | 49 | 56 | 77 | 350 | 175 | 70 | 112 | 105 | 98 | 140 | 91
"Ben's next job," explained Mrs. Barnes, "is an easy and important one. He needs to learn how to multiply by 10."
"I know that already," Ben said. "It's so easy I can't understand why it's so important."
"You can find that out for yourself in the following examples," Mrs. Barnes promised.
14 x 14 = 196
15 x 13 = 195
12 x 16 = 192
16 x 16 = 256
17 x 15 = 255
14 x 18 = 252
18 x 18 = 324
19 x 17 = 323
20 x 16 = 320
17 x 17 = 289
16 x 18 = 288
15 x 19 = 285
Sum 80
120
160
140
6 x | 8 | 7 | 6 | 9 | 10 | 20 | 13 | 25 | 12 | 16 | 14 | 11 | 15
Product | 48 | 42 | 36 | 54 | 60 | 120 | 78 | 150 | 72 | 96 | 84 | 66 | 90
29


30
6 x | 6 | 8 | 10 | 12 | 14 | 16 | 7 | 9 | 11 | 13 | 15 | 20 | 25
Product | 36 | 48 | 60 | 72 | 84 | 96 | 42 | 54 | 66 | 78 | 90 | 120 | 150
Mrs. Barnes said, "We have found that there are several ways to get a correct answer."
"Today, we are going to complete three Cross Number Puzzles. Then, we will tackle these problems in several ways. We will use Ben's, Ruth's, and Anne's methods,"
13 x 17 = 221
12 x 18 = 216
11 x 19 = 209
Sum 180
13 x 17 = (+3) x (10+7) = 100 + 70 + 30 + 21 = 221
(10+3) x(10+7) 21 70 30 +100 221
(Notice:- Ruth changes Ben's regrouping)
Anne's method: 13 x 17 9 130 221
(10 × 13 = 130)
12 x 18 = (+5) x (9+9) = 63 + 63 + 45 + 45 = 216
11 x 19 = (+8) x (9+10) = 27 + 30 + 72 + 80 = 209
Ruth changes regrouping
(9+9) x(4+8) 72 72 36 36 216
(10+9) x(8+3) 80 72 30 27 209
(Ruth changes order)
Anne:
18 x 12 36 180 216
19 x 11 19 190 209
Compare the different methods
8 x | 7 | 9 | 11 | 6 | 8 | 10 | 13 | 15 | 12 | 14 | 16 | 25 | 50
Product | 56 | 72 | 88 | 48 | 64 | 80 | 104 | 120 | 96 | 112 | 128 | 200 | 400
30

There are five in each row—two boys and three girls. In each column there are six children; two look for the product of 11 and 15, and four look for the product of 12 and 14. And in this way, the puzzle points clearly to the answers to the questions above since each cell records the number of boys or girls that work to find each of the two products. The *ears* or circles record the number of students in each row (5) and the number of students in each column (6).

And, again, we see how helpful it is to illustrate the abstract mathematics of Ben's Double-Do-It-to-Both Rule with a real-life situation and a picture of that situation. The importance of this development becomes more apparent on pages 29 through 31. Here, with the introduction of a simple limitation—that, in certain instances, multiplies of 10 are called for and, in others, numbers less than 10—we move, with deeper understanding, close to the traditional algorism of multiplication.

Each development of Ben's story helps build an understanding that each factor can be regrouped freely and of the implications of such regrouping. A sense of the almost limitless freedom begins to emerge in the Double-Do-It-To-All-Three examples on pages 23 and 25, in which one factor is regrouped as a pair of addends and the other as a group of 3 addends. This anticipates such examples as the following, in which the regrouping is based on the convenience of decimal notation:



Since division is simply the inverse of multiplication, it is not surprising that Ben has a division version of the Do-It-to-Both Rule . . . which begins to be revealed on *Work Text* page 26.

Ben does not remember that 5 × 7 = 35, so

$35 \div 5 =$ ____ is not simply a matter of memory. He knows that $25 = 5 \times 5$ and that $10 = 2 \times 5$ and that $35 = 25 + 10$, so he regroups 35 as $25 + 10$ and looks at the problem as:

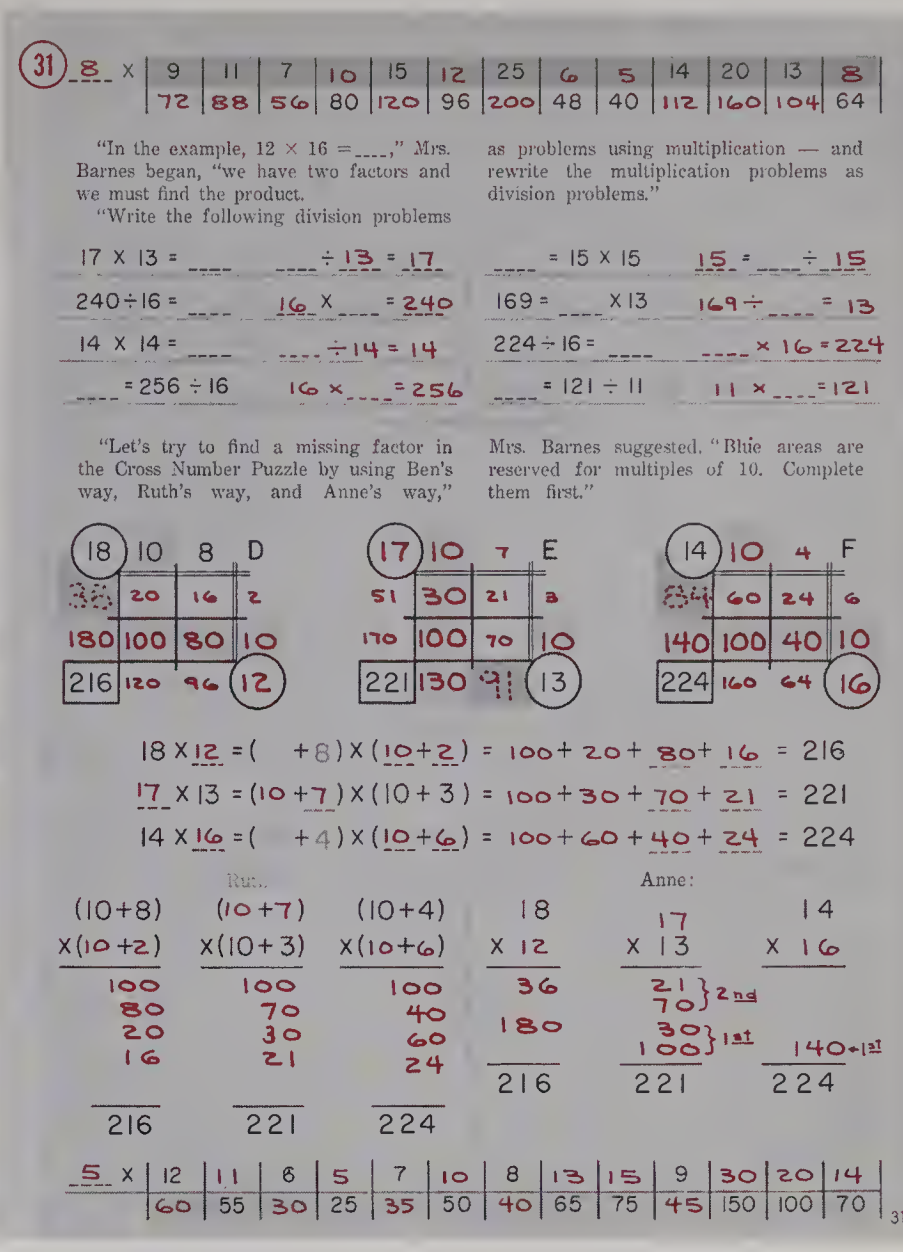
$$(25+10) \div 5 = 25 \div 5 + 10 \div 5 = 5 + 2 = 7, \text{ or}$$

$$\overline{)35} = \overline{5)25 + 10}\ \overset{5+2=7}{} \qquad \text{or} \qquad \overline{)35} = \overline{5)20 + 15}\ \overset{4+3=7}{}$$

A child must be convinced that such manipulations by Ben are reasonable or he will never fully understand the traditional procedure (or algorism) of division – long or short!

Page 27 considers the inverse of Ben's Double-Do-It-to-Both Rule. His division rule helps him compute $56 \div 8 =$ ____. Again, he must do each step because of his limited knowledge of multiplication facts, which we do when we compute $621 \div 27$ because we are limited in our memory of multiplication facts. To follow Ben's reasoning and to be convinced by it is almost a prerequisite to understanding what we call long division.

We return to this same situation on page 31. Here, a Cross Number Puzzle is used to tackle a situation in which the product and one factor are given, but one factor is missing.



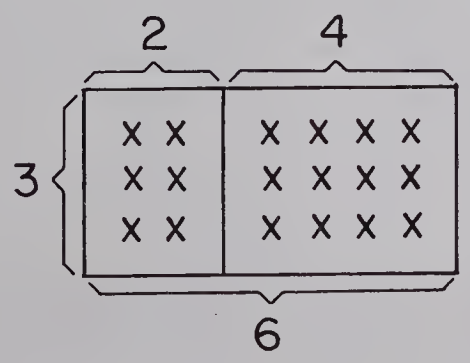
Because the blue areas are reserved for multiples of 10, the rules require a kind of reasoning that comes very close to selecting a reasonable *trial divisor* (as it has been called traditionally).

The idea of *division* is thereafter considered only occasionally until *Work Text* page 84. At this point, the focus is once again placed on the idea of finding the missing factor through a series of regrouping games, and developed much closer to the traditional algorism of long division.

Ben's story goes straight to the principle upon which an understanding of multiplication and division procedures rest. An example, a sketch, and a generalization or pattern-name for this idea are:

example: $3(2 + 4) = 3 \times 2 + 3 \times 4$

sketch:



Generalization: $a(b+c) = ab+ac$ or $a \times (b+c) = a \times b + a \times c$

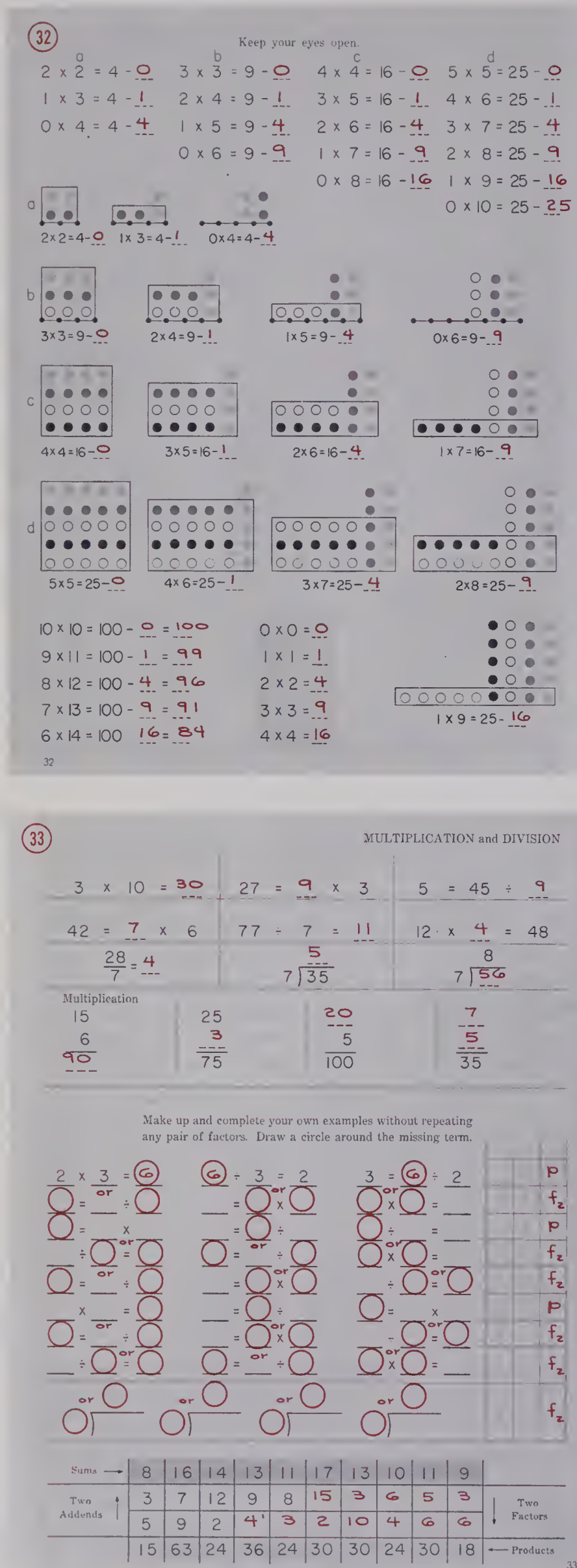
Notes to accompany *Work Text* page 32

The idea given here will not be mastered at this point, only introduced. It is seen again on *Work Text* page 37, is much more fully developed on pages 112-115 of and will be reintroduced at higher levels of the *Math Workshop*.

The idea can be expressed in an algebraic form in this way: If a and b are any numbers you think of (a larger than b, for convenience here), then:

$$(a+b) \cdot (a-b) = a \cdot a - b \cdot b \text{ or } a^2 - b^2$$
$$6 \cdot 4 = (5+1) \cdot (5-1) = 5 \cdot 5 - 1 \cdot 1 = 25 - 1 = 24$$
$$101 \cdot 99 = (100+1) \cdot (100-1) = 100 \cdot 100 - 1 \cdot 1 = 10{,}000 - 1 = 9{,}999$$
$$102 \cdot 98 = (100+2) \cdot (100-2) = 100 \cdot 100 - 2 \cdot 2 = 10{,}000 - 4 = 9{,}996$$
$$103 \cdot 97 = (100+3) \cdot (100-3) = 100 \cdot 100 - 3 \cdot 3 = 10{,}000 - 9 = 9{,}991$$

The reason that this rather amazing idea works is illustrated by the sketches on this *Work Text* page. One effective way to present this page is to ask the children to work it out in small groups (two to four each) without any prior discussion – except an emphasis on the suggestion at the top of the page: "Keep your eyes open" for a useful pattern.



The teacher might also introduce the activity by putting the following examples on the board and asking, "How many of you can complete these sentences in your heads, without paper and pencil?"

$$19 \times 21 = \_\_\_\_$$
$$32 \times 28 = \_\_\_\_$$
$$101 \times 99 = \_\_\_\_$$
$$85 \times 95 = \_\_\_\_$$
$$23 \times 17 = \_\_\_\_$$
$$36 \times 44 = \_\_\_\_$$
$$992 \times 1008 = \_\_\_\_$$

There will probably be no volunteers, so tell the class, "If you keep your eyes open as you work through page 32, you may discover something that will enable you to complete each of those sentences in a few seconds!" That will be sufficient motivation for most children.

The approach outlined on this page will help many look at those brain busters as:

$$(20+1) \cdot (20-1) = 400 - 1 = 399$$
$$(30+2) \cdot (30-2) = 900 - 4 = 896$$
$$(100+1) \cdot (100-1) = 10{,}000 - 1 = 9{,}999$$
$$(90-5) \cdot (90+5) = 8{,}100 - 25 = 8{,}075$$
$$(20+3) \cdot (20-3) = 400 - 9 = 391$$
$$(40-4) \cdot (40+4) = 1{,}600 - 16 = 1{,}584$$
$$(1000-8) \cdot (1000+8) = 1{,}000{,}000 - 64 = 999{,}936$$

Here is certainly a pattern that pays handsome dividends as a shortcut in a large number of examples of multiplication. It will greatly simplify the work on the top half of *Work Text* page 35.

Specific notes are not considered necessary for *Work Text* pages 33-34

Notes to accompany *Work Text* page 35

Refer back to the pattern hunt on *Work Text* page 32. The two top sections of page 35 can be handled as simple exercises with the pattern that was unfolded earlier.

In addition, examples such as $33 \times 27 = \_\_\_\_$ are intermingled with easier examples, such as $3 \times 7 = \_\_\_\_$. When the pattern we are studying is applied to such familiar questions as $2 \times 8 = \_\_\_\_$ and produces sensible results, a feeling of confidence in the pattern is increased.

$$2 \times 8 = (5-3) \cdot (5+3) = 25 - 9 = 16$$

Now, what is the secret hidden in the bottom section of page 35? There is a shortcut to finding

ADDENDS — SUMS — FACTORS — PRODUCTS

Let's agree on the meaning of four simple words: ADDEND, SUM, FACTOR, and PRODUCT.

Since addition and subtraction are two different ways of talking about the same idea, we will describe the operations with the words 'addends' and a 'sum.' Multiplication and division will be described in terms of 'factors and a product'.

Addition and Subtraction:

3 + 4 = 7	8 + 12 = 20	15 + 6 = 21
16 − 9 = 7	13 = 25 − 12	22 − 16 = 6

Addition:

18 6 24	19 7 26
27 13 40	33 21 54

Subtraction:

33 8 25	31 6 25
24 10 14	29 11 18

13 + 7 = 20	1. a_1 a_2 S
25 = 10 + 15	2. S a_1 a_2
25 − 8 = 17	3. S a_1 a_2
39 − 9 = 30	4. a_1 a_2 S
24 + 20 = 44	5. a_1 S a_2
0 = 18 − 18	6. a_1 a_2 S

Addition:

I	II	III
7 9 16	18 6 24	0 14 14

Subtraction:

IV	V	VI
20 9 11	13 6 7	30 8 22

I	a_1	a_2	S
II	a_1	S	a_2
III	a_2	S	a_1
IV	a_1	S	a_2
V	a_2	S	a_1
VI	a_1	a_2	S

Sums →	7	13	14	11	12	17	10	4	12	10	14	10
Addends ↑ and Factors ↓ (other than "1")	3	7	9	8	5	9	3	2	9	5	11	6
	4	6	5	3	7	8	7	2	3	5	3	4
Products →	12	42	45	24	35	72	21	4	27	25	33	24

You have already worked out all the facts printed in green.
Have you learned a secret that makes the rest very easy?

15 × 15 = 225 − 0 = 225	16 × 16 = 256 − 0 = 256	17 × 17 = 289 − 0 = 289
14 × 16 = 225 − 1 = 224	15 × 17 = 256 − 1 = 255	16 × 18 = 289 − 1 = 288
13 × 17 = 225 − 4 = 221	14 × 18 = 256 − 4 = 252	15 × 19 = 289 − 4 = 285
12 × 18 = 225 − 9 = 216	13 × 19 = 256 − 9 = 247	14 × 20 = 289 − 9 = 280
11 × 19 = 225 − 16 = 209	12 × 20 = 256 − 16 = 240	13 × 21 = 289 − 16 = 273
10 × 20 = 225 − 25 = 200	11 × 21 = 256 − 25 = 231	12 × 22 = 289 − 25 = 264

19 × 21 = ___ × ___ − ___ = 399	3 × 7 = ___ × ___ − 4 = 21	
11 × 9 = ___ × ___ − 1 = 99	12 × 8 = 10 × 10 − 4 = 96	
13 × 11 = 12 × 12 − 1 = 143	18 × 22 = 20 × 20 − 4 = 396	
31 × 29 = 30 × 30 − 1 = 899	14 × 10 = ___ × 12 − 4 = 140	
7 × 13 = ___ × ___ − 9 = 91	6 × 14 = ___ × 10 − 16 = 84	
2 × 8 = ___ × 5 − 9 = 16	24 × 16 = 20 × 20 − 16 = 384	
17 × 23 = 20 × 20 − 9 = 391	26 × 34 = 30 × 30 − 16 = 884	
33 × 27 = 30 × 30 − 9 = 891	8 × 16 = ___ × 12 − 16 = 128	

If you know the following facts, there are many others you can do in your head.

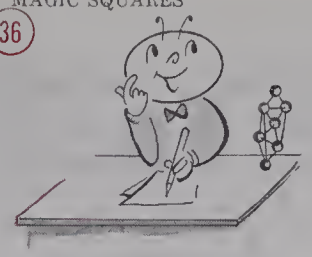
25 × 25 = 625 35 × 35 = 1225 45 × 45 = 2025 55 × 55 = 3025

(In your spare time, study the underlining in color. There is a secret hidden there. When you discover the secret, test it on 65 × 65.)

24 × 26 = ___ × ___ − 1 = 624	53 × 57 = 55 × 55 − 4 = 3021	
36 × 34 = ___ × 35 − 1 = 1224	22 × 28 = 25 × 25 − 9 = 616	
44 × 46 = ___ × 45 − 1 = 2024	21 × 29 = 25 × 25 − 16 = 609	
23 × 27 = 25 × 25 − 4 = 621	20 × 30 = ___ × ___ − 25 = 600	
47 × 43 = 45 × 45 − 4 = 2021	38 × 32 = 35 × 35 − 9 = 1216	
33 × 37 = 35 × 35 − 4 = 1221	39 × 31 = 35 × 35 − 16 = 1209	
54 × 56 = 55 × 55 − 1 = 3024	40 × 30 = ___ × ___ − 25 = 1200	

MAGIC SQUARES

36



A

			27
			27
			27
27	27	27	27

B

41	27	37	105
31	35	39	105
33	43	29	105
105	105	105	105

C

75	5	55	135
25	45	65	135
35	85	15	135
135	135	135	135

D

90	6	66	162
30	54	78	162
42	102	18	162
162	162	162	162

E

116	32	92	240
56	80	104	240
68	128	44	240
240	240	240	240

F

15	1	11	27
5	9	13	27
7	17	3	27
27	27	27	27

G

108	24	84	216
48	72	96	216
60	120	36	216
216	216	216	216

H

29	8	23	60
14	20	26	60
17	32	11	60
60	60	60	60

I

8	8	8	24
8	8	8	24
8	8	8	24
24	24	24	24

J

27	6	21	54
12	18	24	54
15	30	9	54
54	54	54	54

K

145	40	115	300
70	100	130	300
85	160	55	300
300	300	300	300

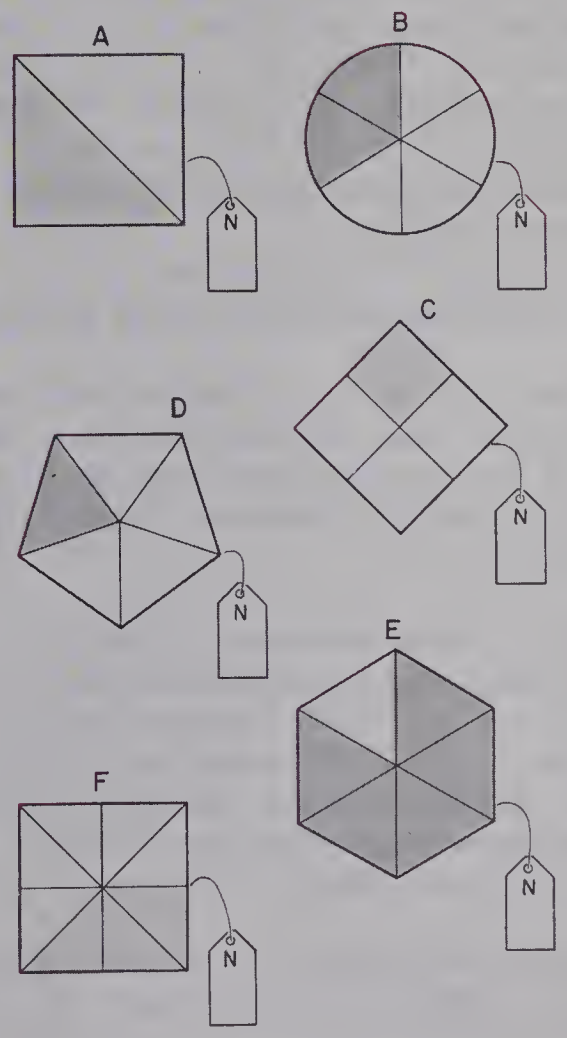
L

135	30	105	270
60	90	120	270
75	150	45	270
270	270	270	270

M

120	8	88	216
40	72	104	216
56	136	24	216
216	216	216	216

37 "N" on the tags means the number suggested by the whole shape.



	Shaded	Un-shaded	Tag
A	3	3	6
B	3	6	9
C	3	9	12
D	3	12	15
E	15	3	18
F	3	5	8
A	4	4	8
B	4	8	12
C	5	15	20
D	4	16	20
	10	20	30
	6	18	24
	50	10	60
	9	15	24
	50	50	100

the product when any number with "5" in the digits column is taken twice as a factor. Let the children work many examples. Together, they will certainly find this shortcut.

25	35	45	55	65
×25	×35	×45	×55	×65
625	*1225*	*2025*	*3025*	*4225*

Obviously, the results always end in 25.

Perhaps a game of "What's My Rule?" will help point to the shortcut:

2,2 → 6	6,6 → 42
3,3 → 12	7,7 → 56
4,4 → 20	8,8 → ____
5,5 → 30	9,9 → ____

Of course, the rule is:

2,2 → 2(2+1) or 2 · 3 or 6
3,3 → 3(3+1) or 3 · 4 or 12
4,4 → 4(+1) or 4 · 5 or 20

Discovery of the pattern is not a main object at this point and should be delayed until it is considered again if it is not discovered at this point. The general pattern is not necessary for completion of the examples, since all depend on the examples that are given at the bottom section on this page.

If one combines a knowledge that $25 \times 25 = 625$ with the pattern already developed on page 32, then:

$$24 \times 26 = (25-1) \cdot (25+1) = 625 - 1 = 624$$

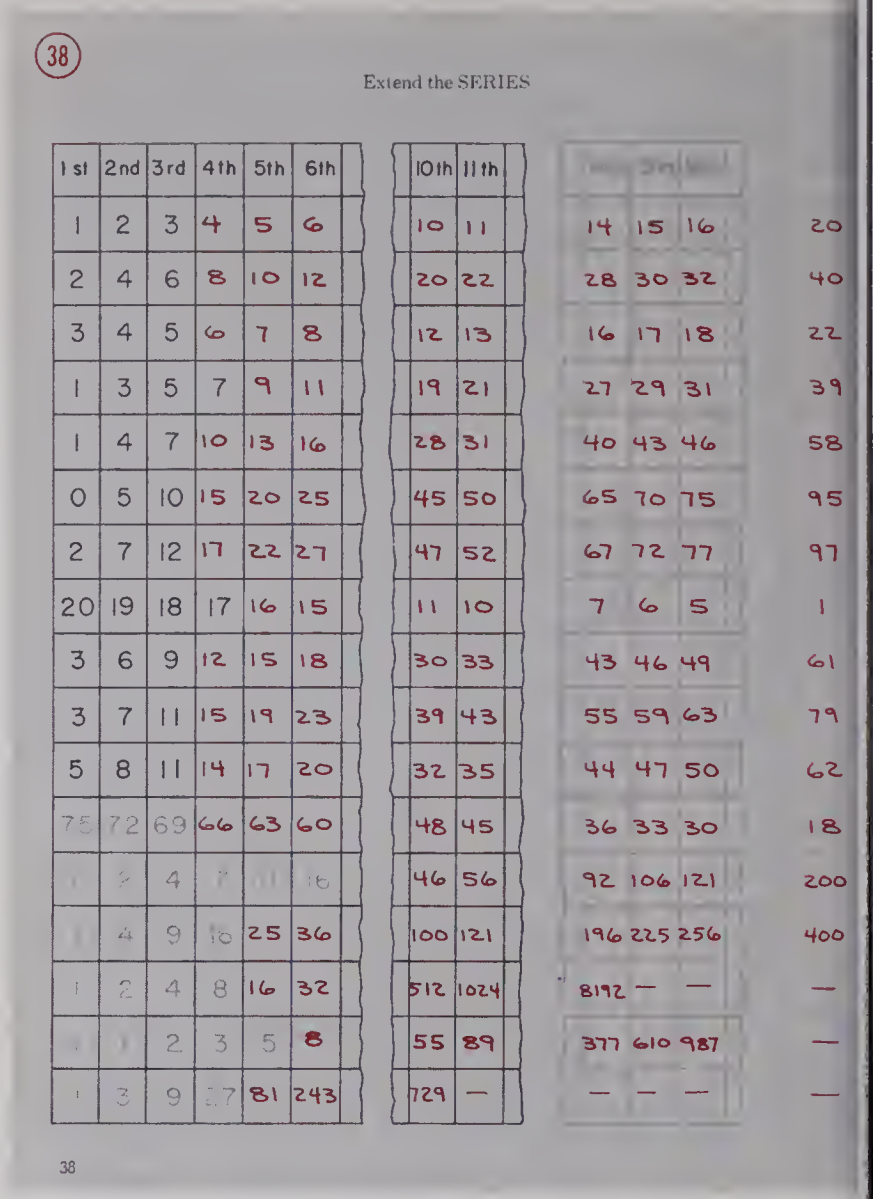
In this fashion, the shortcut found on page 32 can be extended for wider use.

Notes to accompany *Work Text* page 36

Magic squares have often been used to camouflage drill, and that is their purpose here. However, a new element has been added. "Do magic squares preserve their peculiar characteristics when they are operated on?"

Example A is *magic*—the sum of the numbers in all rows, columns, and main diagonals is 27. "Suppose we add 26 to the number written in each of the 9 cells. What happens?" The result is example B. The sum of the three numbers noted in each row, column and main diagonal is 105. Of course, this could have been anticipated, since:

15 + 1 + 11 = 27 and (15+26) + (1+26) + (11+26)
= 15 + 1 + 11 + 3(26) = 27 + 78 = 105



Example C is a study of the results of multiplying the number noted in each cell of A by 5. Now, each row, column, and diagonal will be (5×15) + (5×1) + (5×11) or 5 × (15+1+11) or 5 × 27 or 135. So, in B, C and D, the magic property is preserved.

Carry the operations further. Example E is a study of what happens when the numbers noted in corresponding cells of B and C are added. Example F considers the results of subtracting the number noted in each cell of C from the corresponding cell of D, and the results are again magic. Many operations on magic squares are studied on page 36 and, in each, the magic property is preserved.

An alert class might consider the following: "We have operated on our original magic square in many ways, and it has not lost its magic property. Can you think of other operations that would preserve this property? Can you think of any that would destroy the property?"

"Multiply each number by itself—square it." (The magic property is destroyed.)

This will lead to many mathematically rich considerations of what operations will and will not affect the magic property. Each assertion can be tested by considering several examples. Thus, while

the focus is on studying an intriguing problem, an unusual amount of exercise in basic facts and algorisms is employed.

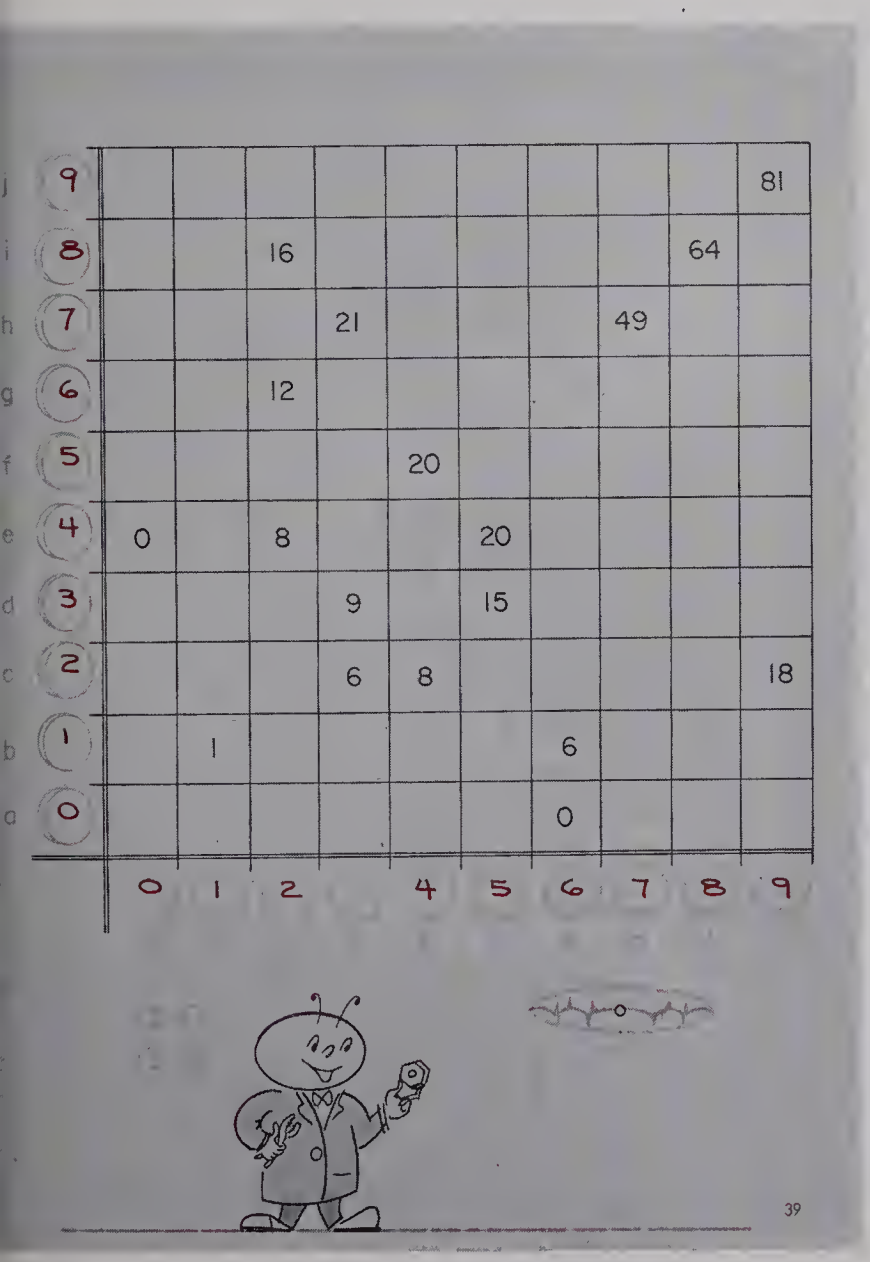
Specific notes are not considered necessary for *Work Text* page 37

Notes to accompany *Work Text* page 38

This page involves another version of "What's My Rule?". Put 1, 2, 3, ____, ____, ____ on the board and ask for volunteers to extend the series by guessing your rule. Few children will have any questions or hesitation.

The same procedure will quickly lead to extension of all the beginnings on this page—except the last five examples. The first two of the last five examples become obvious as the following analysis is made: Each number written in the second and third rows is the difference of the two numbers written above on the right and left.

1 2 4 7 11 16
1 2 3 4 5
1 1 1 1
1 4 9 16
3 5 7
2 2



But this method is no help in the last three examples:

1 2 4 8 16
1 2 4 8
1 2 4

1 1 2 3 5 8 13 21
0 1 1 2 3 5 8
1 0 1 1 2 3

1 3 9 27 81
2 6 18 54
4 12 36
8 24

In the first two examples, the original series is repeated in the second and third rows. In the last, each diagonal down and to the right turns out to be a series in which each term in twice the one above and to the left; and the left diagonal is a repetition of the first of these three examples: 1, 2, 4, 8, etc.

However, by this time, the rules for continuing each of the series has probably been uncovered. In order, the rules are: Each succeeding term is twice its predecessor; each succeeding term is the sum of its two predecessors; each is three times its predecessor.

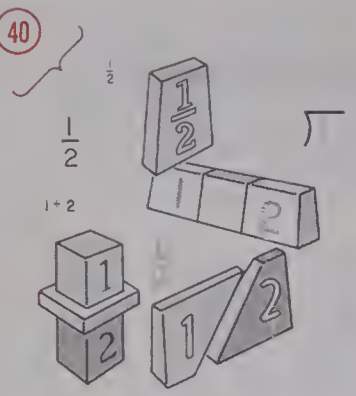
Perhaps the most difficult part of the activity is the challenge of bridging the gaps from the 6th to the 10th term, from the 11th to the 14th, and from the 16th to the 20th.

Many children will write out each series and then record the terms in the appropriate columns. Others may see that a rule can be worked out, based on a relationship to the place in the series.

If n is the number corresponding to the place in the series, then the following patterns would help:

First example: n
Second example: $2 \times n$
Third example: $n + 2$
Fourth example: $2 \times n - 1$
Fifth example: $3 \times n - 2$
Sixth example: $5 \times n - 5$
Seventh example: $5 \times n - 3$
Eighth example: $21 - n$
Ninth example: $3 \times n$
Tenth example: $4 \times n - 1$
Eleventh example: $3 \times n + 2$
Twelfth example: $76 - n$

Specific notes are not considered necessary for *Work Text* page 39



Is a FRACTION a NUMBER?

"My brother Bill said a fraction is a number," said Bobbie. "My sister said a fraction is a pair of numbers. My Aunt thought a fraction was either a numeral or a pair of numerals."

"My sister said that fractions and whole numbers belong to two very different number systems. Bill said she was wrong because you can mix fractions and whole numbers, so they can't be so very different. Now, I'm all mixed up. Who is right?"

Mrs. Barnes asked the class if they could help Bobbie. Not a hand went up.

"Well," Mrs. Barnes said, "even mathematicians disagree about some of those questions. Let's not tackle those problems yet. Rather, let's look a little deeper into fractions."

"First, how do we use the idea of fractions in every day life? Here are some statements that mention fractions. What do they mean?"

a. My uncle gave me a half-dollar.
b. How many eggs are there in a half-dozen eggs?
c. Bill traded me one out of every two of his 36 marbles for my model airplane.
d. I paid 25c for a book. The clerk told me that was one-half of the regular price.
e. When I asked the clerk for a pint of cream, he asked me: "Do you want it in a pint carton or two half-pint cartons?"
f. At the carnival the ticket seller shouted: "Who's next to buy a ticket — only one dime, a mere two nickels, two-fifths of a quarter; yes sir, just one-tenth of a dollar. Who's next?"
g. Jim ran one-third of the way to the store and walked the other four blocks.
h. The swimming pool was 60 feet long. Alice swam two-thirds of the length before she got tired.
i. Pencils were 3 for 10c. I got nine of them.
j. Five girls made two dollars profit on a bake sale. They divided the money evenly.
k. I'll be ready in a quarter of an hour.
l. Because of trouble that developed, they stopped the countdown a half-minute before blast-off time.
m. Uncle Pete gave me and Alice a half-dollar and asked us to divide it evenly between us.
n. Lois got four out of every five problems right. She got 20 right.
o. The plane was three-quarters of an hour late.

Sums →	12	15	16	12	12	14	18	14	18	14	14	
Two Addends ↑	9	7	8	5	8	6	10	9	9	7	3	↓ Two Factors
	3	8	8	7	4	8	8	5	9	7	11	
	27	56	64	35	32	48	80	45	81	49	33	← Products

"Let's take our study of fractions further," Mrs. Barnes suggested. "We know a lot about whole numbers. Let's see if fractions behave in a similar way. Try these tests. They may shed light on the problem."

Mrs. Barnes' TEST ONE: A [illegible] is [illegible] to [illegible] fractions. [illegible]

(whole numbers)		
3, 7, 1, 6, 5	$\frac{8}{2}, \frac{6}{2}, \frac{2}{2}, \frac{5}{2}, \frac{1}{2}$	$\frac{6}{3}, \frac{9}{3}, \frac{3}{3}, \frac{1}{3}, \frac{2}{3}$
1 < 3 < 5 < 6 < 7	$\frac{1}{2} < \frac{2}{2} < \frac{5}{2} < \frac{6}{2} < \frac{8}{2}$	$\frac{1}{3} < \frac{2}{3} < \frac{3}{3} < \frac{6}{3} < \frac{9}{3}$
$\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{8}$	$\frac{3}{3}, \frac{3}{1}, \frac{3}{2}, \frac{3}{6}, \frac{3}{4}$	$\frac{2}{8}, \frac{2}{5}, \frac{2}{2}, \frac{2}{1}, \frac{2}{4}$
$\frac{1}{8} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2}$	$\frac{3}{6} < \frac{3}{4} < \frac{3}{3} < \frac{3}{2} < \frac{3}{1}$	$\frac{2}{8} < \frac{2}{5} < \frac{2}{4} < \frac{2}{2} < \frac{2}{1}$
$\frac{6}{3}, \frac{8}{2}, \frac{4}{4}, \frac{1}{2}, \frac{3}{4}$	$\frac{12}{4}, \frac{1}{8}, \frac{2}{3}, \frac{3}{2}, \frac{8}{1}$	$\frac{3}{2}, \frac{1}{2}, \frac{9}{4}, \frac{8}{4}, \frac{4}{5}$
$\frac{1}{2} < \frac{3}{4} < \frac{4}{4} < \frac{6}{3} < \frac{8}{2}$	$\frac{1}{8} < \frac{2}{3} < \frac{3}{2} < \frac{12}{4} < \frac{8}{1}$	$\frac{1}{2} < \frac{4}{5} < \frac{3}{2} < \frac{8}{4} < \frac{9}{4}$
$\frac{5}{5}, \frac{1}{3}, \frac{6}{8}, \frac{2}{4}, \frac{4}{6}$	$\frac{1}{6}, \frac{3}{4}, \frac{1}{2}, \frac{2}{3}, \frac{1}{8}$	$\frac{3}{6}, \frac{8}{9}, \frac{1}{5}, \frac{4}{7}, \frac{6}{8}$
$\frac{1}{3} < \frac{2}{4} < \frac{4}{6} < \frac{6}{8} < \frac{5}{5}$	$\frac{1}{8} < \frac{1}{6} < \frac{1}{2} < \frac{2}{3} < \frac{3}{4}$	$\frac{1}{5} < \frac{3}{6} < \frac{4}{7} < \frac{6}{8} < \frac{8}{9}$

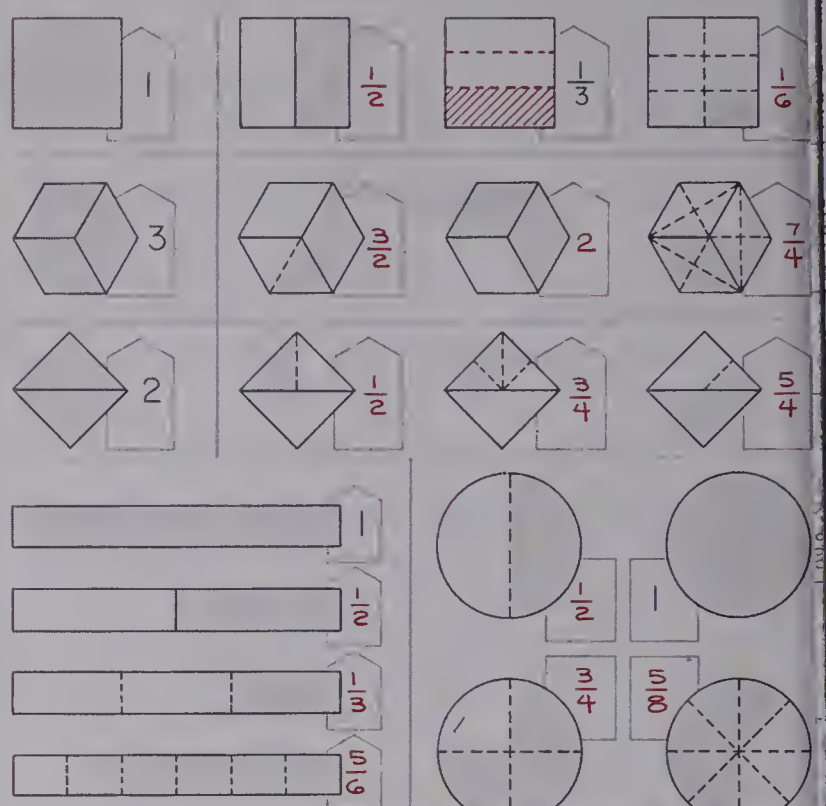
Brain Buster

$\frac{1}{6}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{10}, \frac{1}{3}, \frac{5}{6}, \frac{1}{4} \longrightarrow \frac{1}{10} < \frac{1}{6} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{5}{6}$

Sums	16	48	20	20	48	40		100	28	44	90	
Two Addends ↑	12	40	15	18	42	35	25	75	21	33	80	Products
	4	8	5	2	6	5	5	25	7	11	10	One Factor (not 1)
	3	5	3	9	7	7	5	3	3	3	8	Other Factor

Mrs. Barnes' TEST TWO: Can you draw pictures that suggest the idea for fractions? Can you do it so the pictures include the idea of whole numbers?

If this picture is labeled to suggest a whole number — can you label the rest of the pictures and make sense?



Sums →	16	16	16	16	16	16	5	25	18	4	22	14	20
Addends ↑	12	9	3	2	10	5	2	20	7	2	11	7	10
Factors ↓ (not 1)	4	7	13	14	6	11	3	5	11	2	11	7	10
Products	48	63	39	28	60	55	6	100	77	4	121	49	100

Mrs. Barnes' TEST THREE: Are there any fractions that are other names for whole numbers?

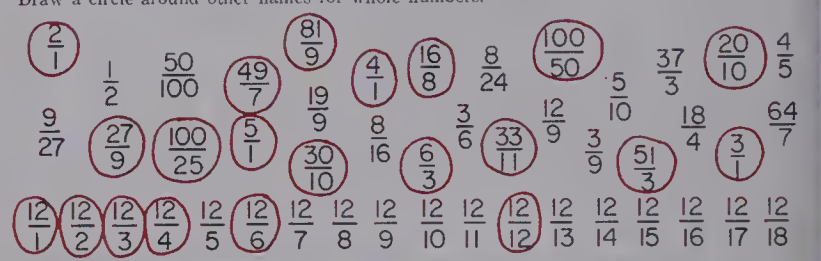
Bill volunteered first. "Two half-dollars is another name for one whole dollar. This suggests the idea that $\frac{2}{2}$ might be another name for the whole number, 1."

"Eight quarters is another name for two dollars," Betty added. "I think $\frac{8}{4}$ is another name for 2."

"Ten half-gallons of milk is the same amount as five gallons of milk," said Jane. And it makes sense to me to say that $\frac{10}{2}$ is another name for 5."

Mrs. Barnes began writing on the board. "Which fraction names whole numbers?"

Draw a circle around other names for whole numbers.



Mr. B[illegible] TEST FOUR

In each of the boxes below, write fractions that are other names for the whole numbers in the blue boxes.

2	5	3	1	4	0
$\frac{2}{1}$	$\frac{5}{1}$	$\frac{3}{1}$	$\frac{1}{1}$	$\frac{4}{1}$	$\frac{0}{1}$
$\frac{4}{2}$	$\frac{10}{2}$	$\frac{6}{2}$	$\frac{2}{2}$	$\frac{8}{2}$	$\frac{0}{2}$
$\frac{6}{3}$	$\frac{15}{3}$	$\frac{9}{3}$	$\frac{3}{3}$	$\frac{12}{3}$	$\frac{0}{3}$
$\frac{8}{4}$	$\frac{50}{10}$	$\frac{12}{4}$	$\frac{9}{9}$	$\frac{20}{5}$	$\frac{0}{99}$
$\frac{10}{5}$	$\frac{100}{20}$	$\frac{30}{10}$	$\frac{67}{67}$	$\frac{40}{10}$	$\frac{0}{1000}$
Etc.	Etc.	Etc.	Etc.	Etc.	Etc.

Make the following into TRUE STATEMENTS:
< means less than, > means greater than, = means equal.

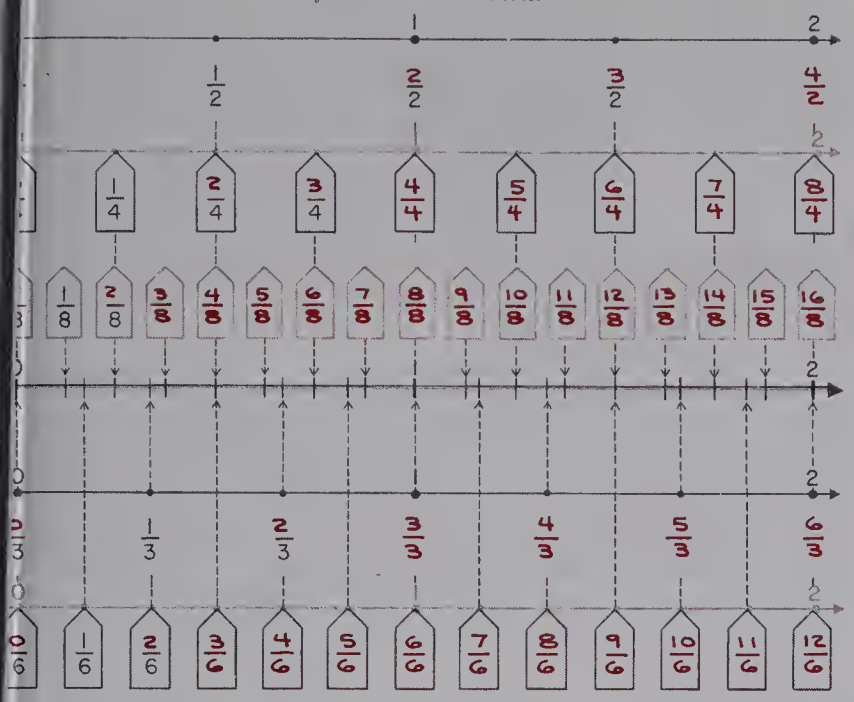
$\frac{2}{1} < \frac{8}{2}$ $\quad \frac{9}{3} = \frac{12}{4}$ $\quad \frac{21}{7} > \frac{14}{7}$ $\quad \frac{50}{10} > \frac{36}{9}$ $\quad \frac{1}{1} = \frac{12}{12}$ $\quad \frac{6}{2} < \frac{28}{7}$

$\frac{55}{11} > \frac{12}{3}$ $\quad \frac{10}{10} = \frac{1}{1}$ $\quad \frac{18}{2} > \frac{24}{3}$ $\quad \frac{1}{1} > \frac{0}{7}$ $\quad \frac{72}{9} < \frac{63}{7}$ $\quad \frac{100}{100} = \frac{77}{77}$

	5	4	12	20	9	0	27	44	17	6	0	← Addend
Factor →	8	12	15	25	15	9	30	50	21	6	0	← Sum
Factor →	3	8	3	5	6	9	3	6	4	0	0	← Addend
Product →	24	96	45	125	90	81	90	300	84	0	0	

Mrs. Barnes' TEST FIVE Can we use fractions as labels for points on the number line?

Fill in the tags below so that they make sense as labels.



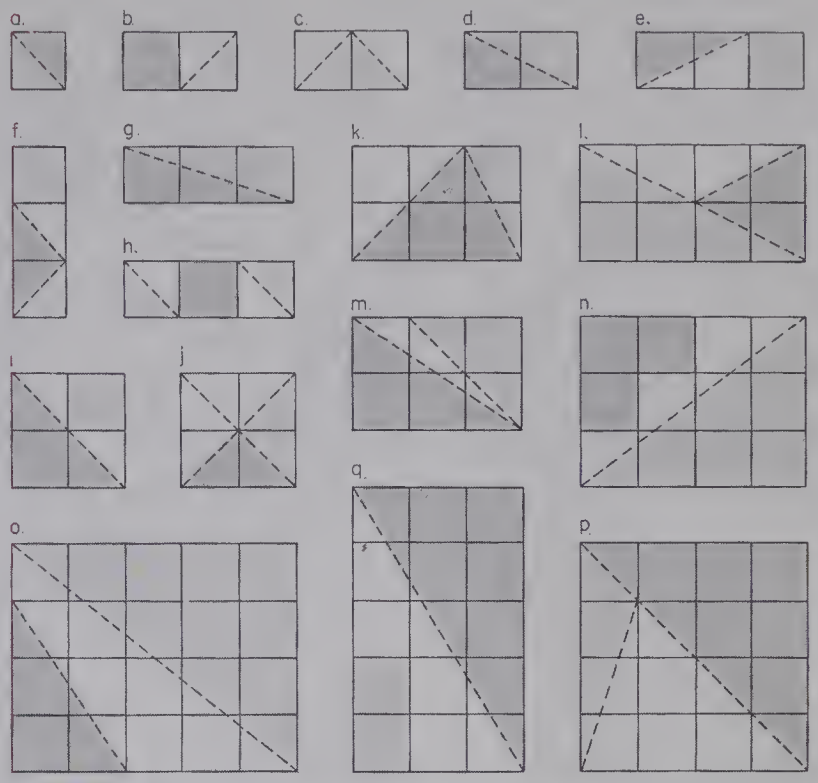
Using $=$, $>$ or $<$, make the following true.

$\frac{1}{2} < \frac{3}{4}$	$\frac{2}{6} = \frac{1}{3}$	$\frac{7}{6} > \frac{8}{8}$	$\frac{8}{6} < \frac{11}{8}$	$\frac{5}{3} > \frac{3}{5}$	$\frac{7}{8} > \frac{5}{6}$
$\frac{1}{4} = \frac{2}{8}$	$\frac{4}{6} < \frac{3}{4}$	$\frac{12}{6} = \frac{16}{8}$	$\frac{6}{8} > \frac{4}{6}$	$\frac{7}{4} > \frac{10}{6}$	$\frac{7}{4} = \frac{14}{8}$
$\frac{1}{6} > \frac{1}{8}$	$\frac{6}{6} = \frac{4}{4}$	$\frac{9}{8} < \frac{8}{6}$	$\frac{1}{4} < \frac{2}{6}$	$\frac{3}{2} = \frac{9}{6}$	$\frac{3}{6} = \frac{2}{4}$

12x →	2	3	5	4	9	10	8	18	11	7	30	50	20
Product	24	36	60	48	108	120	96	216	132	84	360	600	240

then what numbers are suggested by the following:

— area shaded green?
— area shaded blue?
— area not shaded?
— area shaded blue, plus area shaded green?
— area shaded blue, plus unshaded area?
— area shaded green, plus unshaded area?



	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
area shaded in green suggests	$\frac{1}{2}$	1	0	1	1	1	$1+\frac{1}{2}$	1	2	1	3	2	3	3	3	8	$7+\frac{1}{2}$
area shaded in blue suggests	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	$1+\frac{1}{2}$	1	1	1	1	4	2	6	10	2	2
unshaded area suggests	0	$\frac{1}{2}$	1	0	1	1	0	1	1	2	2	2	1	3	7	6	$5+\frac{1}{2}$
total area suggests	1	2	2	2	3	3	3	3	4	4	6	8	6	12	20	16	15

"I noticed something on the last page that may always be true. If two fractions name the same point on the number line, then I can cross-multiply and get the same number." Mrs. Barnes asked Ben to give the class some examples.

Example I

$\frac{24}{3\times8} = \frac{24}{4\times6}$

$\frac{3}{4} = \frac{6}{8}$

Before Ben went on Mrs. Barnes explained that "$\neq$" means "does not equal".

Example II

$12 \neq 14$

$\frac{4}{7} \neq \frac{2}{3}$

Example III

$25 \neq 24$

$\frac{5}{6} \neq \frac{4}{5}$

Let's test Ben's rule with more examples.

$8 \neq 9$	$6 = 6$	$42 \neq 40$	$40 = 40$	$36 = 36$
$\frac{2}{3} \neq \frac{3}{4}$	$\frac{2}{6} = \frac{1}{3}$	$\frac{7}{8} \neq \frac{5}{6}$	$\frac{5}{10} = \frac{4}{8}$	$\frac{9}{6} = \frac{6}{4}$
$48 = 48$	$70 \neq 72$	$18 = 18$	$49 \neq 48$	$24 = 24$
$\frac{3}{4} = \frac{12}{16}$	$\frac{7}{8} \neq \frac{9}{10}$	$\frac{2}{3} = \frac{6}{9}$	$\frac{7}{12} \neq \frac{4}{7}$	$\frac{12}{8} = \frac{3}{2}$

"I think Ben could go further," Anne suggested. "He could have put signs for 'greater than' and 'less than' — '>' and '<' — in the lower blue box. Which ever sign makes that statement below true, will also make the statement above it true . . . I think." "Is that right?" Mary wanted to know. "Let's do more experimenting." was Mrs. Barnes answer to Mary's question.

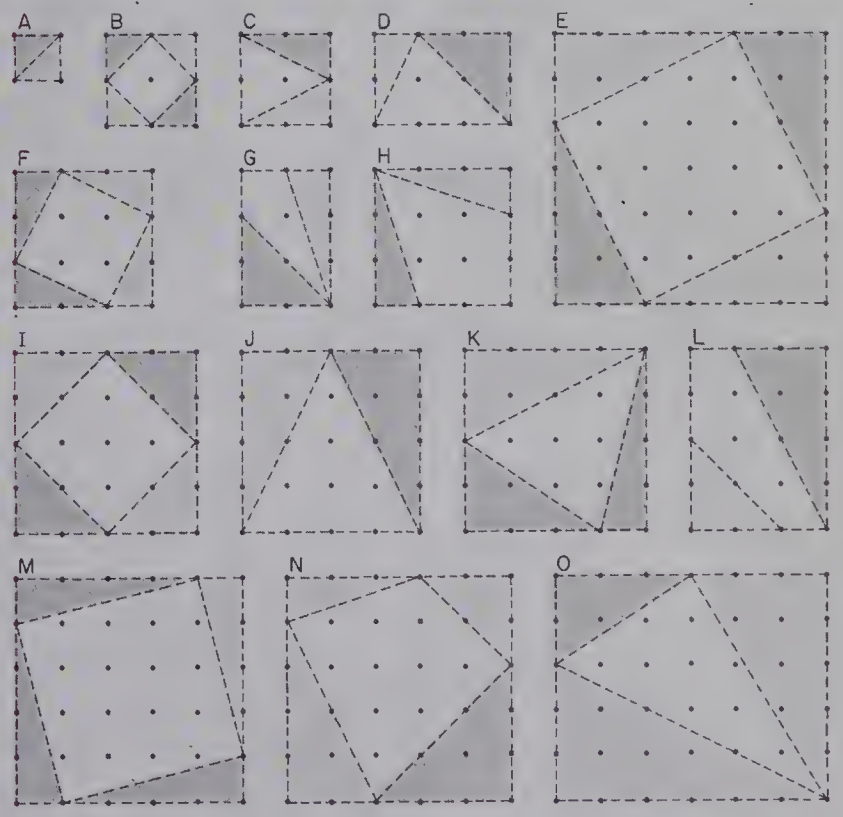
$8 < 9$	$9 > 4$	$7 < 9$	$49 < 81$	$28 = 28$
$\frac{2}{3} < \frac{3}{4}$	$\frac{3}{2} > \frac{2}{3}$	$\frac{1}{9} < \frac{1}{7}$	$\frac{7}{9} < \frac{9}{7}$	$\frac{7}{14} = \frac{2}{4}$
$24 = 24$	$54 < 56$	$27 < 28$	$75 > 72$	$300 = 300$
$\frac{3}{4} = \frac{6}{8}$	$\frac{6}{8} < \frac{7}{9}$	$\frac{3}{4} < \frac{7}{9}$	$\frac{5}{9} > \frac{8}{15}$	$\frac{25}{100} = \frac{3}{12}$

Let's see what happens if we work backwards.

$21 < 22$	$15 < 16$	$24 = 24$	$35 < 36$	$48 < 49$
$\frac{3}{11} < \frac{2}{7}$	$\frac{5}{2} < \frac{8}{3}$	$\frac{12}{3} = \frac{8}{2}$	$\frac{7}{4} < \frac{9}{5}$	$\frac{8}{7} < \frac{7}{6}$

13 X →	2	5	7	10	17								
Products →	26	65	91	130	221								

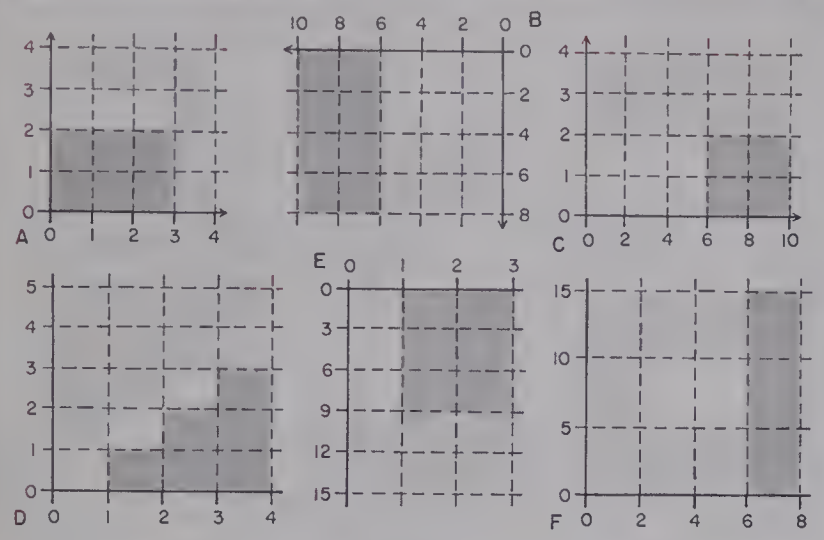
On this page we agree that is 1.



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Area shaded in green suggests	$\frac{1}{2}$	1	1	2	8	2	2	$1+\frac{1}{2}$	4	4	5	4	6	$4+\frac{1}{2}$	3
Area shaded in blue suggests	$\frac{1}{2}$	1	1	1	8	2	$1+\frac{1}{2}$	$1+\frac{1}{2}$	4	4	4	2	2	$7+\frac{1}{2}$	$16+\frac{1}{2}$
Unshaded area suggests	0	2	2	3	20	5	$2+\frac{1}{2}$	6	8	8	7	6	17	13	$10+\frac{1}{2}$
Total area suggests	1	4	4	6	36	9	6	9	16	16	16	12	25	25	30

48

Addend	90	7	80	17	20	85	8	60	10	189
Sum	100	100	90	77	100	100	48	72	200	200
Addend	10	93	10	60	80	15	40	12	190	11

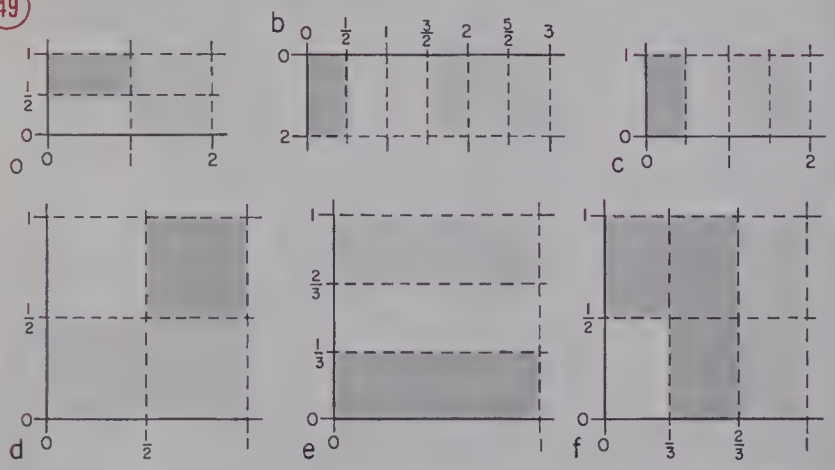


In each picture, the green, blue and unshaded areas suggest:

	□	□	□	□+□	□+□	□+□	□+□+□
A	6	7	3	9	10	13	16
B	32	24	24	56	48	56	80
C	8	20	12	20	32	28	40
D	6	10	4	10	14	16	20
E	18	18	9	27	27	36	45
F	30	70	20	50	90	100	120

Sum	20	21	25	33	30	21	40	40	22	35	42	102
Addend	2	16	10	4	28	4	33	17	6	24	5	95
Addend	18	5	15	29	2	17	7	23	16	11	37	7

49

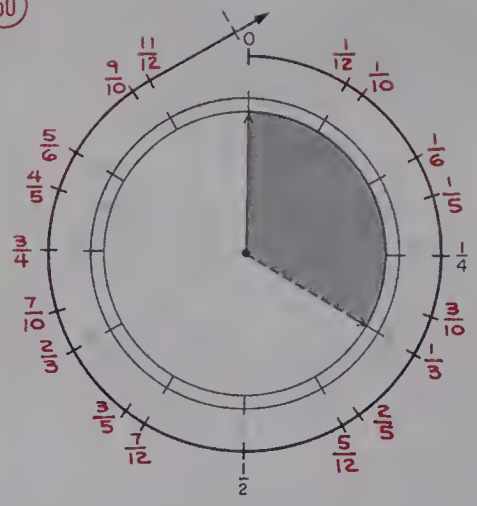


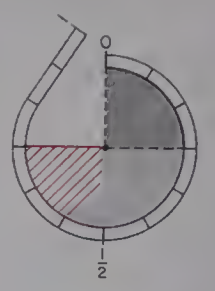
	□	□	□	□+□	□+□	□+□	□+□+□
a	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$1+\frac{1}{2}$	$1+\frac{1}{2}$	2
b	$\frac{1}{2}$	$1+\frac{1}{2}$	1	$\frac{1}{2}+1$	$1+\frac{1}{2}+1$	$1+\frac{1}{2}+\frac{1}{2}$	3
c	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$1+\frac{1}{2}$	$1+\frac{1}{2}$	2
d	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	1
e	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1
f	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{6}$	1

x	6	2	7	5	8	9
8	48	16	56	40	64	72
9	54	18	63	45	72	81
3	18	6	21	15	24	27
4	24	8	28	20	32	36

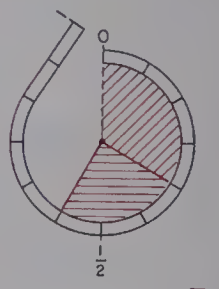
+	10	18	14	17	12
8	18	26	22	25	20
6	16	24	20	23	18
7	17	25	21	24	19
9	19	27	23	26	21

50

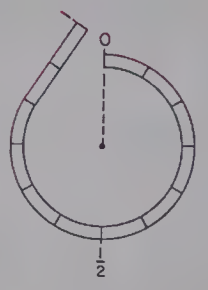




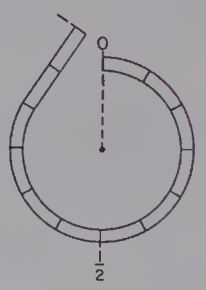
a. $\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$



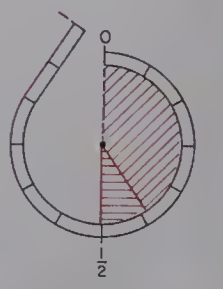
b. $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$



d. $\frac{1}{12}+\frac{1}{3}=\frac{5}{12}$



e. $\frac{5}{12}+\frac{1}{6}=\frac{7}{12}$



c. $\frac{2}{5}+\frac{1}{10}=\frac{1}{2}$

f. $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$

g. $\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$

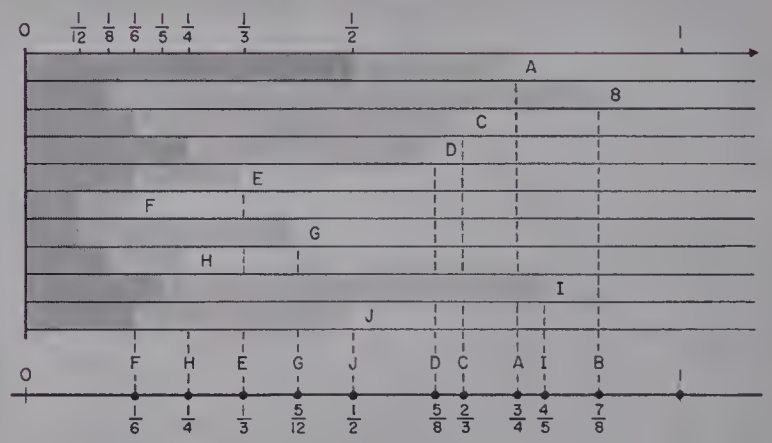
h. $\frac{2}{3}+\frac{1}{12}=\frac{3}{4}$

i. $\frac{2}{3}+\frac{1}{4}=\frac{11}{12}$

j. $\frac{1}{12}+\frac{1}{4}=\frac{1}{3}$

k. $\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$

51



A. $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ B. $\frac{1}{8}+\frac{3}{4}=\frac{7}{8}$ C. $\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$ D $\frac{1}{4}+\frac{3}{8}=\frac{5}{8}$

E. $\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$ F. $\frac{1}{12}+\frac{1}{12}=\frac{1}{6}$ G. $\frac{1}{12}+\frac{1}{3}=\frac{5}{12}$ H $\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$

I. $\frac{1}{5}+\frac{3}{5}=\frac{4}{5}$ J. $\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$ K $\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$ L. $\frac{5}{8}+\frac{1}{4}=\frac{7}{8}$

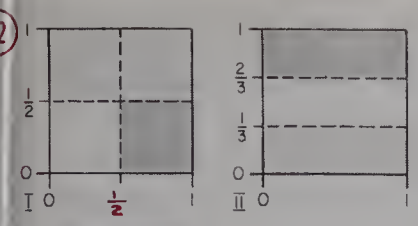
Another way of looking at the same pictures:

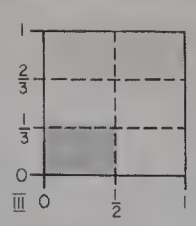
A. $\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$ B $\frac{1}{8}=\frac{7}{8}-\frac{3}{4}$ C $\frac{2}{3}-\frac{1}{6}=\frac{1}{2}$ D $\frac{1}{4}=\frac{5}{8}-\frac{3}{8}$

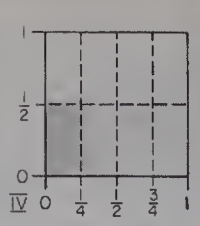
E. $\frac{1}{3}-\frac{1}{6}=\frac{1}{6}$ F $\frac{1}{12}=\frac{1}{6}-\frac{1}{12}$ G $\frac{1}{3}=\frac{5}{12}-\frac{1}{12}$ H $\frac{1}{4}-\frac{1}{8}=\frac{1}{8}$

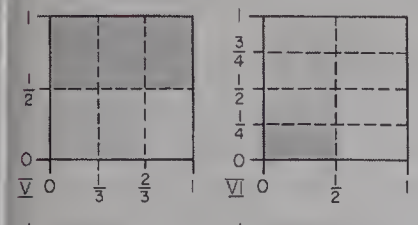
I $\frac{4}{5}-\frac{3}{5}=\frac{1}{5}$ J $\frac{1}{3}=\frac{1}{2}-\frac{1}{6}$ K. $\frac{5}{6}-\frac{2}{3}=\frac{1}{6}$ L $\frac{1}{4}=\frac{7}{8}-\frac{5}{8}$

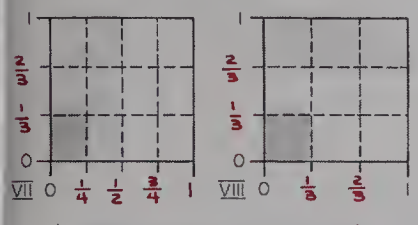
	14	14	10	16	17	15	20	25	29	52	38	Sum
Factor	12	1	7	7	17	8	10	20	25	50	36	Addend
Factor	2	13	3	9	0	7	10	5	4	2	2	Addend
Product	24	13	21	63	0	56	100	100	100	100	72	

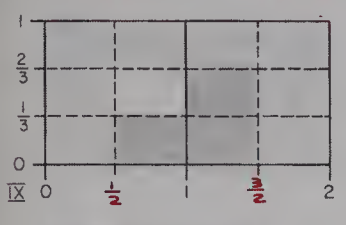


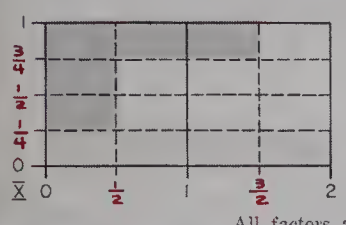












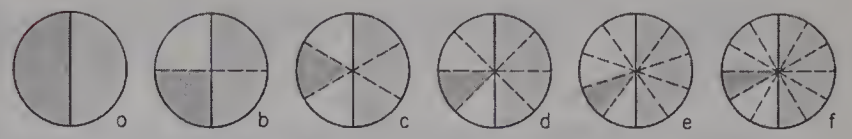
	□	□	□	□ + □	□ + □
I	1/2	1/4	1/4	3/4	1/2
II	1/3	1/3	1/3	2/3	2/3
III	1/3	1/6	1/2	1/2	2/3
IV	3/8	1/8	1/2	1/2	5/8
V	1/3	1/2	1/6	5/6	2/3
VI	5/8	1/8	1/4	3/4	3/8
VII	5/12	1/12	1/2	1/2	7/12
VIII	1/3	1/9	5/9	4/9	2/3
IX	4/3	1/2	1/6	11/6	2/3
X	1/2	5/8	7/8	9/8	3/2

All factors are greater than two!

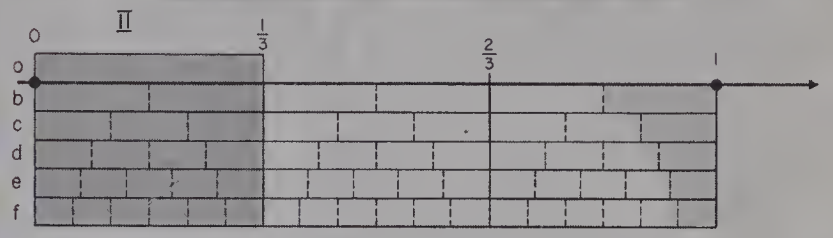
Product	21	49	27	42	63	81	35	39	45	56
Factor	3	7	3	6	7	9	5	3	5	8
Factor	7	7	9	7	9	9	7	13	9	7

If fractions are numbers, they must have many names!

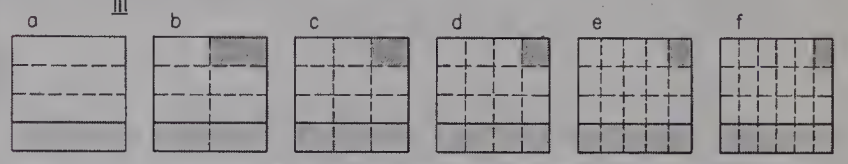
I



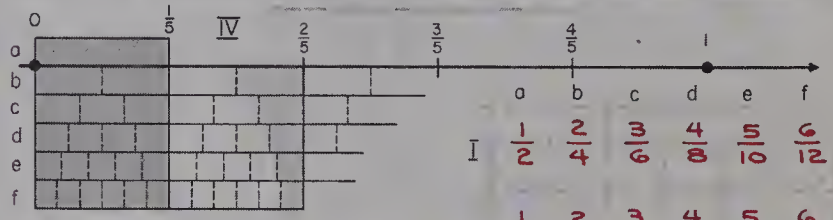
II



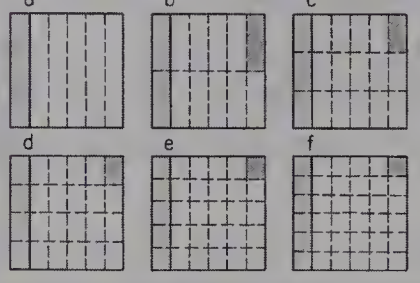
III



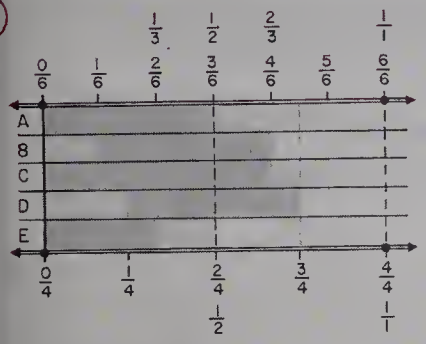
IV



V



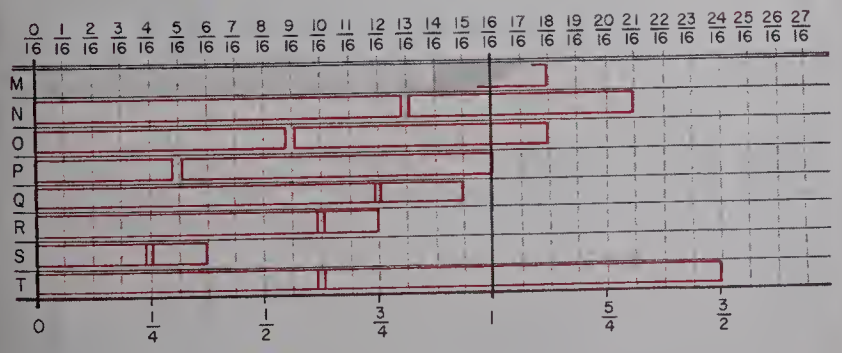
	a	b	c	d	e	f
I	1/2	2/4	3/6	4/8	5/10	6/12
II	1/3	2/6	3/9	4/12	5/15	6/18
III	1/4	2/8	3/12	4/16	5/20	6/24
IV	1/5	2/10	3/15	4/20	5/25	6/30
V	1/6	2/12	3/18	4/24	5/30	6/36



A. $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ B. $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ G. $\frac{2}{8} + \frac{3}{12} = \frac{1}{2}$ H. $\frac{3}{8} + \frac{1}{4} = \frac{5}{8}$

C. $\frac{2}{3} + \frac{1}{6} = \frac{5}{6}$ D. $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ I. $\frac{1}{12} + \frac{1}{2} = \frac{7}{12}$ J. $\frac{1}{4} + \frac{5}{8} = \frac{7}{8}$

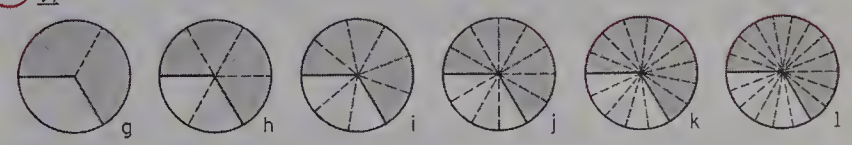
E. $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ F. $\frac{2}{6} + \frac{2}{3} = \frac{1}{1}$ K. $\frac{3}{4} + \frac{1}{6} = \frac{11}{12}$ L. $\frac{7}{12} + \frac{5}{12} = \frac{4}{4}$



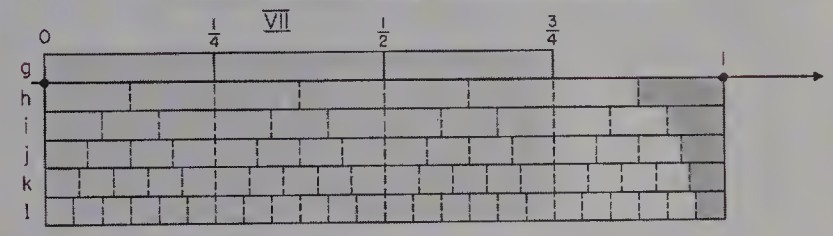
M. $\frac{7}{8} + \frac{1}{4} = \frac{9}{8}$ N. $\frac{13}{16} + \frac{1}{2} = \frac{21}{16}$ O. $\frac{9}{16} + \frac{9}{16} = \frac{18}{16}$ P. $\frac{5}{16} + \frac{11}{16} = \frac{16}{16}$

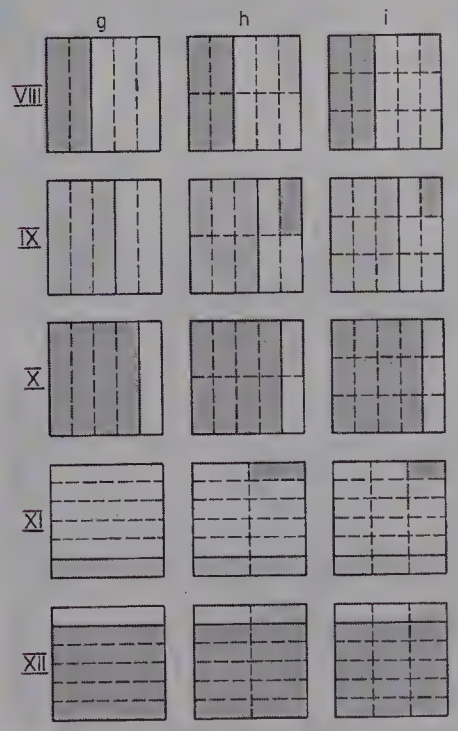
Q. $\frac{3}{4} + \frac{3}{16} = \frac{15}{16}$ R. $\frac{5}{8} + \frac{1}{8} = \frac{3}{4}$ S. $\frac{4}{16} + \frac{1}{8} = \frac{3}{8}$ T. $\frac{5}{8} + \frac{7}{8} = \frac{3}{2}$

VI



VII





	g	h	i	j	k	l
VI	2	4	6	8	10	12
	3	6	9	12	15	18
VII	3	6	9	12	15	18
	4	8	12	16	20	24
VIII	2	4	6	8	10	12
	5	10	15	20	25	30
IX	3	6	9	12	15	18
	5	10	15	20	25	30
X	4	8	12	16	20	24
	5	10	15	20	25	30
XI	1	2	3	4	5	6
	6	12	18	24	30	36
XII	5	10	15	20	25	30
	6	12	18	24	30	36

Notes to accompany *Work Text* pages 40-55

Pages 40 through 55 (and continuing on through 67) explore some of the ideas of fractions. This study is an extension of our study of number from whole numbers to those numbers which men invented so that there would be an answer to every division problem. Without fractions, $2 \div 3 =$ ____ and $3 \div 2 =$ ____ have no meaning; there is no way to complete these mathematical sentences.

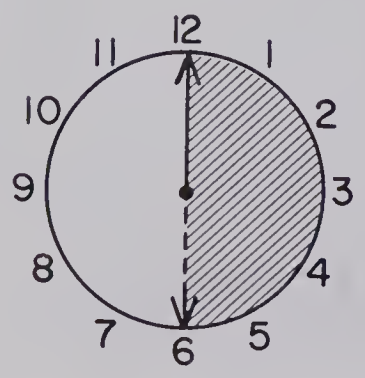
The approach to fractions used in the *Math Workshop for Children* is to consider this topic not as a separate idea, but rather as an extension of the idea of number. Fractions are explored on *Work Text* pages 40 to 44 to see if they behave like the numbers we know. We examine them to find the answers to these questions:

1) Can fractions be arranged from small to large?
2) Can pictures be drawn to suggest the idea in the context of a picture that also suggests whole numbers?
3) Are there fractions that are other names for whole numbers?
4) Is there at least one fraction to name each whole number?
5) Can fractions help name points on the number line?
6) Can fractions be added? can they be subtracted?
7) Can fractions be multiplied and divided?

Fraction activities

In the material that follows the various expressions used in speaking of fractions will be studied briefly. No rules or procedures should be discussed until pupils have first had an opportunity to acquaint themselves with the whole body of knowledge regarding fractions. Before rules or procedures can be meaningful, this knowledge must be worked into sensible and understandable patterns that are consistent with all the arithmetic of whole numbers pupils have already learned.

Activity 1. Draw a clock on the board, with the minute hand pointing to 12. "Where will the minute hand be in half-hour?" Have a volunteer draw an indication of the hand's second position.



"Color in the area the minute hand covered as it traveled from the first to the second position. How much of the clock face is colored? Have we divided the clock face into two parts about the same size?"

Start over with the minute hand pointing to 12. "Where will the minute hand be in a quarter of an hour? Where will it be a quarter of an hour later? Indicate both positions. Where will it be half an hour from the starting time? Where will it be three-quarters of an hour from the starting time? Indicate this new position. We have shown four of the positions the minute hand passes. Color in the part of the face the minute hand passes over in a quarter of an hour." (If colored chalk is available:) . . . "with another colored chalk, shade another part of the face that the hand would pass over in a half-hour."

Follow a similar discussion with one-third and two-thirds of an hour; also with one-sixth, two-sixths, three-sixths, four-sixths, five-sixths of an hour.

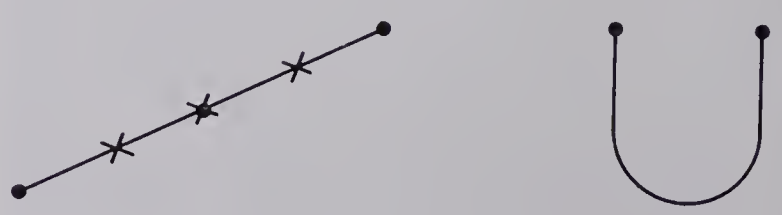
Activity 2. Put a square on the board. "Can someone draw a line that divides the square into 2 parts about the same size? Can someone else draw a different line that divides the square into 2 parts? Any more lines?" (Any straight line drawn through a point where the two diagonals would cross is a correct response. Some child may draw a zig-zag or curved line that would have to be accepted.)

Activity 3. Draw a line on the board—preferably at a slight angle—and put a dot on each end.

"This is my picture of a piece of wire. The ends are sharp and I don't want to get scratched, so I've covered them. Where should I cut the wire to make two pieces the same length? Mark the point with a dot.

"Suppose I wanted to cut the wire into four pieces that are the same size. Can you mark with a cross the places where I ought to cut it?

"Now I'm going to draw another sketch of my piece of wire. This time I will bend it a little bit.

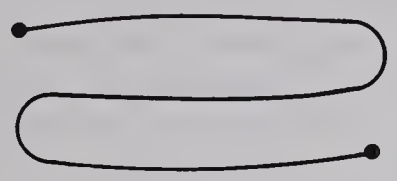


"Where should I cut the wire if I wanted to have two pieces the same size? Can you mark with a dot the place where I should cut it? And where would you put cross marks to indicate where I should cut the wire if I want to have four pieces the same size?

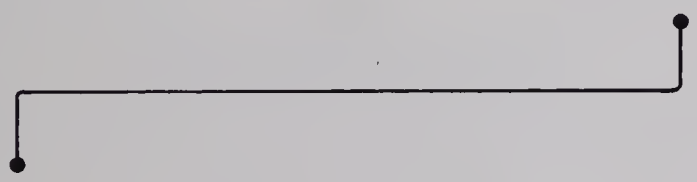
"I found a longer piece of wire."

"Where should I cut it, if I want to have two pieces of the same size? Where should I cut it, if I want to have four pieces the same size?

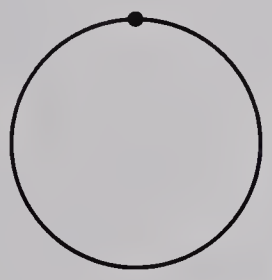
"Now I've bent the wire into an S-shape.



"Now I put two sharp bends in it.

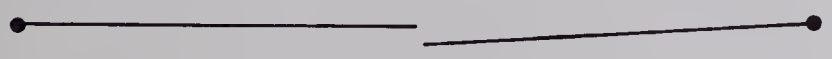


"Finally, I bend it into something like a circle.



"In each instance, mark with a dot the place where I should cut the wire if I want to have two pieces the same size, and then mark with crosses the places where I should cut if I want to have four pieces the same size.

"I finally cut my wire into two pieces.



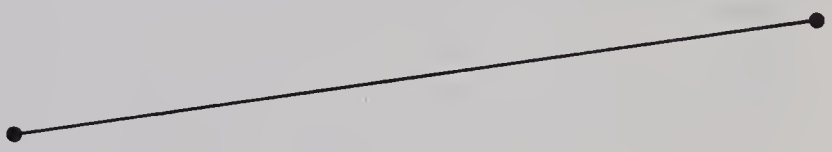
"No sooner had I done this than I was sorry. I had really wanted to cut it into four pieces that were the same size . . . but I only have two. What can I possibly do?" (You will have plenty of advice.)

"If I change my mind again after I have cut the wire in four pieces, is there anything I can do?" (You'll have more advice: solder two pieces, twist them together, glue them, nothing, get another piece of wire, etc.)

We have carefully avoided using the word "equal." This expression is, as previously explained, to be used in referring to the relationship between two names for the same number: $3 + 9 = 12$. Besides, in dealing with wires or sketches, it is not possible to cut them or divide them into "equal" pieces. If we remind pupils that what we really mean when we say "of the same size" is "*about* the same size," there won't be any misunderstanding. "Remember, when we are concerned with answering a question of 'How many,' we must always be exact. But when we are concerned with answering a question that involves any kind of measurement and that asks 'How much,' we can never be exact."

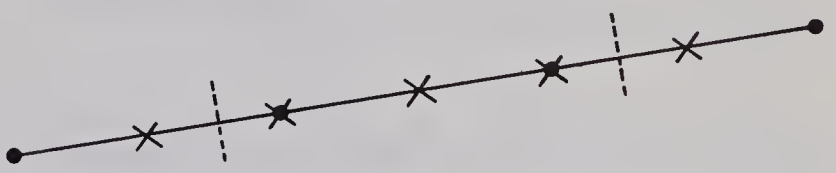
Because the kind of sketches used in this activity are meant only to *suggest* fractions, answers need only be approximate.

Activity 4. "I have a rope. I've knotted each end so it won't unravel. If I were to lay it out straight, it would look something like this:



"If I wanted to cut the rope into three pieces about the same size, where should I cut it? Mark the places with dots.

"Now suppose I wanted to have twice as many pieces of rope, all about the same size. Using a little cross can you mark each place on the rope where I should cut it?



"Why do some of the dots and crosses mark the same place?" (Don't expect well formulated answers. Don't try to settle the matter.)

"If I wanted to have only two pieces about the same size, where would I cut? Is the place marked?

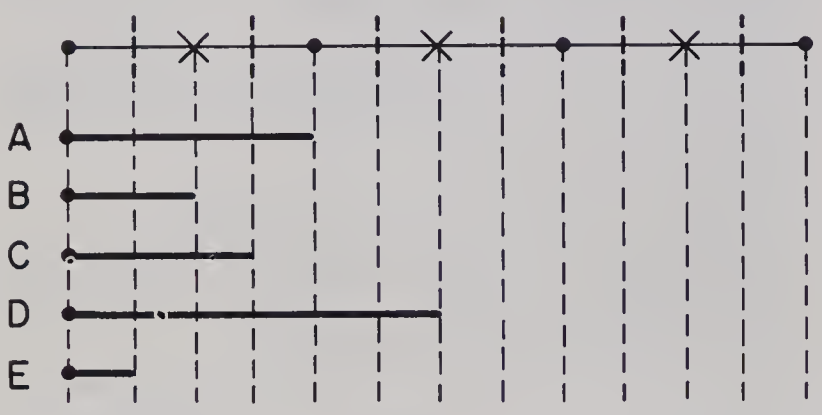
"Suppose I wanted four pieces about the same size? (Cut it in half—and then cut each of the two pieces in half.) Are all the places to be cut marked? Use a dotted line to mark the new places I would need to cut.

"Now, what do you suppose has happened? I've

changed my mind again. I want a dozen pieces – all about the same size. Where shall I cut?" Discuss this matter for a while. You might cut the rope in half and then cut each of those pieces in half to get four pieces. You would then cut each of the four pieces into three parts. Or you might cut the rope into six parts; and cut each of the six pieces in half. Or, you might cut the rope into three parts; and then cut each of those in half, and halve each of the six pieces; etc.

"Are any of the places to be cut already marked? How many more places do we have to mark?" (4) "Indicate these places with dotted lines."

On the board draw the following:



"At one time, I wanted to cut my rope in two pieces about the same size. Does any line on the chart look something like one of those pieces?" (The line marked D.)

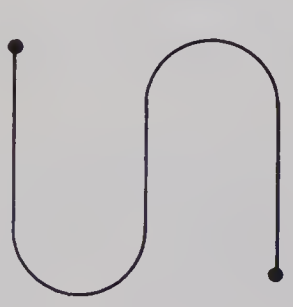
"What do the other sketches suggest to you?

"How many pieces of rope the size of line B would I be able to cut from a piece of rope the size of line D?

"Is the line marked A twice as long as the line marked C? Which line is twice as long as A?"

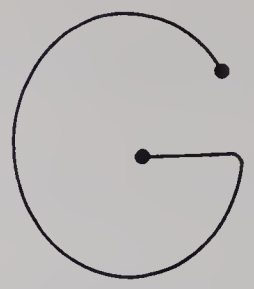
Shift the discussion to a pattern similar to that outlined for page 88 by "bending or twisting your rope."

"Now, I've twisted my rope until it looks like a letter S turned over on its side:



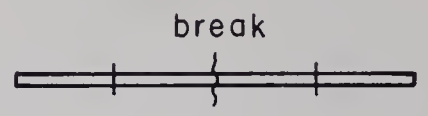
"Use dots to mark the places on the rope that should be cut, if I want to have three pieces about the same size. Next, use little crosses to show where I should cut the rope if I want six pieces about the same size." Do not go further with this example.

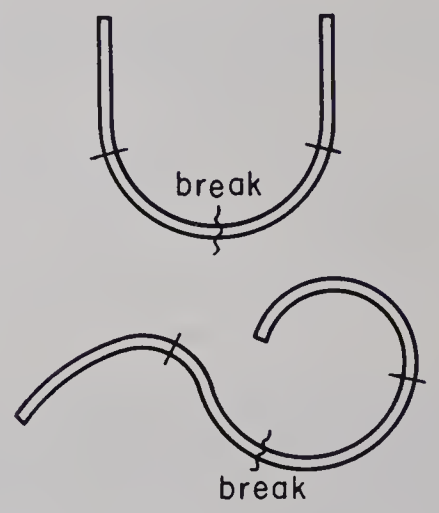
"Now, I have twisted the rope into the shape of the letter G."



Have the children mark the places to be cut if you want three and then six pieces.

Activity 5. "Let's talk about some taffy that Martha made. She pulled it into a long piece and layed it on the kitchen table. In what shape do you think she put it down? Well, she might have layed it out straight . . . or in the shape of the letter U . . . or in a variety of other shapes. Here are some possibilities:



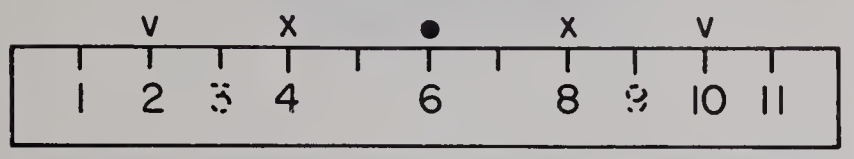


"Martha decided that she wanted to divide the candy into three parts: one for Billy, one for Jean and one for herself. So she marked the places she would break it when the candy got hard." Have pupils make the markings.

"Then Martha remembered that Billy had braces on his teeth and couldn't eat taffy. So, she decided to break it in half when it was hard. She marked the place. "Have a volunteer mark the place." When the candy was hard, Martha broke the large piece into smaller ones. Just then Billy came running into the kitchen. 'Look, Martha, no more braces. Oh, boy, you've made some taffy.' "Now what do

you think Martha did?" She could break each half into three pieces each so that she would have six pieces: two for Billy, two for Jean and two for herself. Or, she could break the taffy at the first marks she made and have two large pieces and two small pieces. Now are the two small pieces together about the same size as one of the large pieces?

Activity 6. Here is a picture of a one-foot ruler.



"Where is the middle of the ruler? What number is usually written at that point on a one-foot ruler? Use small crosses to divide the length of the ruler into three parts about the same size. What numbers would be written at these points?"

"What points would divide the ruler into six parts about the same size? Label the points as they usually are labeled on a ruler."

"Is it easy to divide the ruler into 4 parts about the same size? Label these points if they are not already labeled."

"Are there any more points to be labeled on this ruler?"

Activity 7. "In the shorthand of arithmetic, there are at least three different ways of writing the following sentence." On the board, write: *Six divided by two equals three.*

"How many ways can you write this statement in shorthand, using no written words at all?" The three ways that interest us are given below with 2 examples of each:

$6 \div 2 = 3$ $\qquad \frac{6}{2} = 3$ $\qquad 2\overline{)6}$

$12 \div 4 = 3$ $\qquad \frac{12}{4} = 3$ $\qquad 4\overline{)12}$

This suggests that the following statements are true:

$$6 \div 2 = 12 \div 4 \text{ and } \frac{6}{2} = \frac{12}{4} \text{ and } 2\overline{)6} = 4\overline{)12}$$

In each case, read these sentences: "Six divided by 2 equals 12 divided by 4."

"The shorthand $\frac{6}{2} = \frac{12}{4}$ suggests a game. The rules are simple. Make all of the following sentences true, using nothing but this shorthand in which to express division facts. We never use the numeral 1 unless we have to."

$\frac{9}{3} = \frac{12}{?}$ $\qquad \frac{16}{4} = \frac{?}{7}$ $\qquad \frac{25}{?} = \frac{15}{3}$

$\frac{9}{3} + \frac{4}{2} = \frac{20}{?}$ $\qquad \frac{15}{5} + \frac{12}{3} = \frac{?}{2}$

If there is any question about these examples, write them each three ways. For instance:

$9 \div 3 = 12 \div ?$ and $\frac{9}{3} = \frac{12}{?}$ and $3\overline{)9} = ?\overline{)12}$

$(9 \div 3) + (4 \div 2) = 20 \div ?$

$\frac{9}{3} + \frac{4}{2} = \frac{20}{?}$

$3\overline{)9} + 2\overline{)4} = ?\overline{)20}$

After pupils have grasped the idea of the game, move on to examples for which there are several solutions:

$\frac{18}{3} - \frac{18}{2} = \frac{?}{7}$ (because $6 - 4 = 2$);

$\frac{9}{?} + \frac{8}{?} = \frac{?}{2}$ (because $3 + 2 = 5$);

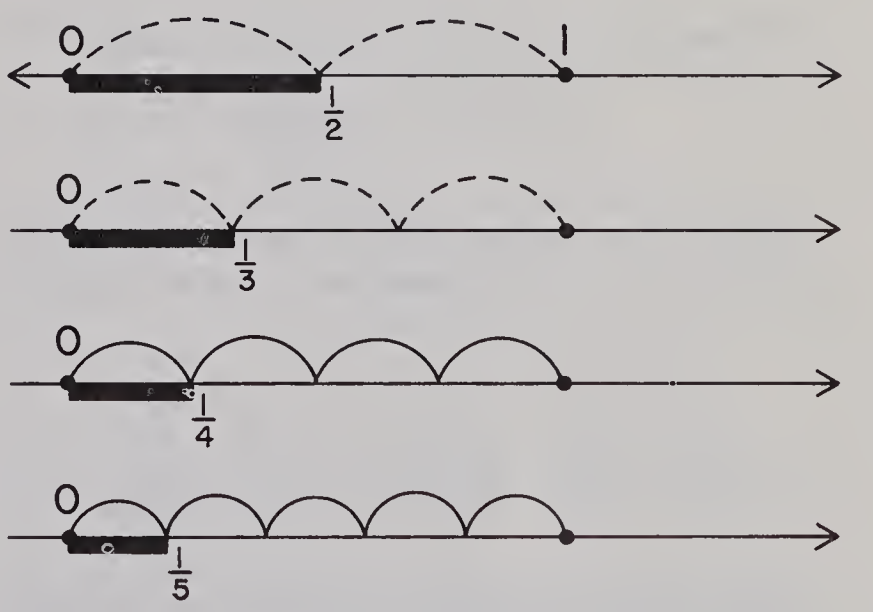
$\frac{?}{2} + \frac{?}{3} = \frac{?}{8}$; $\frac{8}{2} \times \frac{6}{3} = \frac{?}{6}$;

$\frac{20}{2} \div \frac{6}{3} = \frac{?}{5}$; etc.

As pupils play this game, they are on the firm and familiar ground of arithmetic. They will be asked to play the game often. And when they finally come to: $\frac{2}{3} + \frac{3}{4} = ?$ and $\frac{1}{3} \times \frac{2}{5} = ?$, pupils will not find these expressions mysterious. They will have a feeling that such expressions probably fit into the scheme of: $\frac{8}{2} + \frac{6}{3} = \frac{54}{9}$

Activity 8. "Indicate jumps on a number line that suggest;"

$1 \div 2$ or $1/2$	$1 \div 6$ or $1/6$
$1 \div 3$ or $1/3$	$1 \div 7$ or $1/7$
$1 \div 4$ or $1/4$	$1 \div 8$ or $1/8$
$1 \div 5$ or $1/5$	$1 \div 9$ or $1/9$



Have the children complete an example suggesting each of the expressions shown above. When confronted with an example such as $1 \div 4$ or $1/4$, discuss procedures that might be followed, such as: first find the point halfway between 0 and 1; then find the point halfway between 0 and this new point (or,

between 0 and 1/2). When preparing to draw a sketch to suggest the idea of $1 \div 6$ or 1/6, the first step might be the same as in the previous example, or it might be to indicate the points suggesting $1 \div 3$, and then to find the mid-points of each segment. In many cases, the work can begin exactly as for one of the other examples, and then carried further. In such cases, two examples will show their family resemblances. Since 1/6 is related both to 1/2 and 1/3, either of these can be used to begin the work. One-eighth is related in an even closer way to 1/2 and 1/4. One-ninth bears a family relationship to 1/3.

"What labels do not seem to be related to any others in the group under consideration?" (Only 1/5 and 1/7.)

"Do you suppose that 1/5 and 1/7 belong to any families at all?" (Yes, 1/5 is related to 1/10, 1/15, 1/20, etc.; and 1/7 is related to 1/14, 1/21, 1/28, etc.

Refer back to the diagrams suggesting 1/2, 1/3 ... 1/9. "If we want to label all the landing points between 1 and 0, what labels would you suggest in each instance?" (2/3, 1/3; 3/4, 2/4, 1/4; 4/5, 3/5, 2/5, 1/5; etc.)

"Starting from 0, which is larger, a jump to 3/5 or a jump to 1/2?"

"In the following pairs, circle the member that is located further away from 0 on the number line."

1/3, 2/3	3/4, 1/4
2/4, 1/5	2/3, 5/6
2/8, 1/5	1/2, 3/7

"Consider the following. Which is further from 0?"

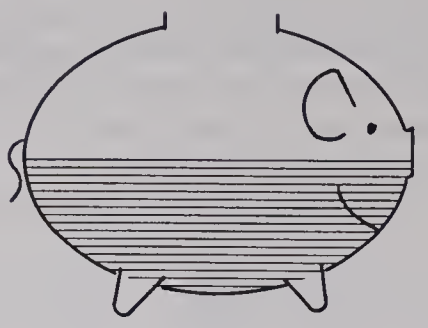
1/2, 3/6	2/4, 1/2
2/6, 1/3	2/3, 6/9

"We have a shorthand that expresses these ideas. I will write some true statements in this shorthand. Can you complete the others?"

$1/2 > 1/3$	$1/2 < 3/4$	$1/2 = 3/6$
$3/9 = 1/3$	$3/6 < 4/6$	$7/8 > 1/4$
2/4 3/6	2/3 2/4	1/7 1/4
1/3 2/7	3/5 1/2	4/8 3/6

Continue examples until everyone can use this notation. Then explain that ">" is shorthand for "is greater than . . ." and "<" is shorthand for "is less than . . ." Point out that if the shorthand symbols reminds one of an arrow, the thin or sharp-end points to the "smaller of the two," and the wide part of the symbol is close to the "larger of the two."

Activity 9. "Here is a sketch of a piggy bank. It is half-full of pennies."

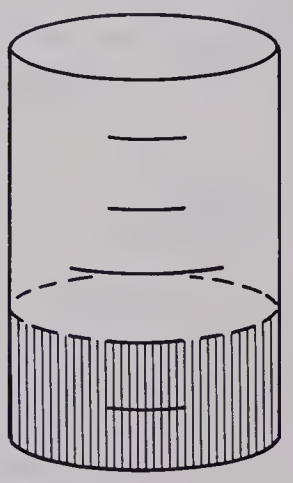


"Can you tell me how many pennies there are? What do you need to know before you can tell me the approximate number? If I told you how many pennies were needed to fill the piggy bank, could you tell me about how many pennies the bank holds when it is half-full?"

Value when Full	No. of Pennies when half-full
$1.00	
$ .50	
$3.00	

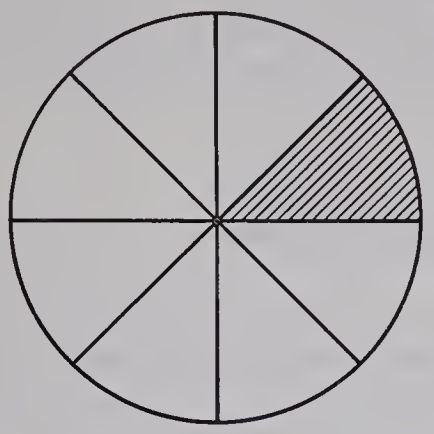
Much of mathematics can be stated in the form of "*If..., then....*" propositions. *If* a full piggy bank holds 150 pennies, *then* about how many pennies does it hold when it is half-full?

Activity 10. "Here is a water tank. The marks on it help to indicate how full it is."



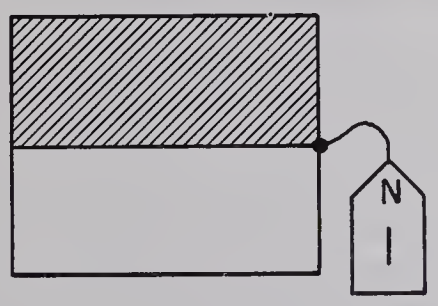
No. of Gallons when full	No. Gallons in shaded area
6	
60	
3	
1	

Activity 11. "Here is a pie that is cut into 8 pieces. There are 16 ounces in a pound. How many ounces would there be in a shaded piece?"



Weight of Whole pie	Weight of 1 piece	
In pounds	In pounds	In ounces
1	1/8	2
2		
4		
6		

Activity 12. "Here is a sketch of 1 of something. This is indicated by the 1 on the tag. What does the shaded area suggest? The unshaded area?"



Shaded	Unshaded	Tag
1/2	1/2	1
		2
		1/2

"Suppose we change the label or tag. Let's write 2 in the tag. Now, what do the shaded and unshaded areas suggest?" (1 and 1)

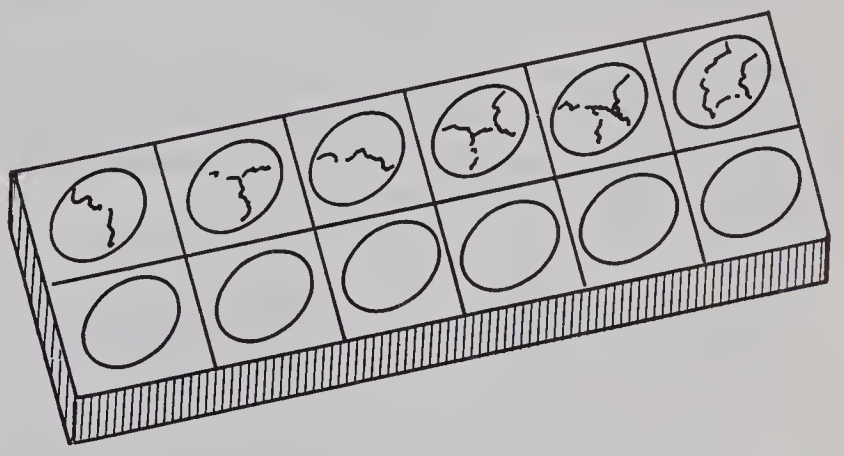
"Suppose we change the tag to read 1/2. Now, what does the shaded area suggest? (1/4 and 1/4) Without discussing, make the comment—"It seems that a half of one-half is one-fourth. In money this would mean that half of a half-dollar is the same as a quarter of a dollar."

"Can you make sketches suggested by each row in the following chart?"

Shaded	Unshaded	Tag
3/4	1/4	1
1/3		1
	2/5	1
3/2		2
	5/3	2

Don't hurry. Discuss the adequacy of each sketch, possible improvements, similar sketches that suggest the same idea, similar ideas suggested by the same sketch, etc.

Activity 13. "Bill said that 6 eggs in a carton of 12 were cracked. His sister said, 'But it looks to me as if 3 in each half-dozen are broken.' His little brother said, 'In each group of two eggs, one is broken.' When his Mother looked at the carton she said, 'Why half of them are broken.' "



"Here is a sketch of the situation. Who was right?" Write on the board:

Bill: "6 eggs are cracked."
Sister: "3 in each half-dozen are broken."
Little brother: "One out of every 2 is broken."
Mother: "Half are broken."

"Are they all saying much the same thing? In mathematics, we say that numbers have many names. We say, for example, that:

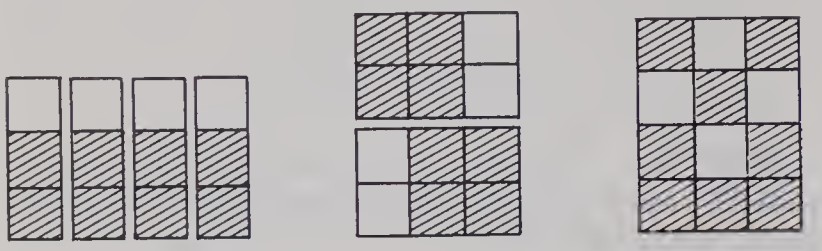
$$\underline{1/2} = 2/4 = \underline{3/6} = 4/8 = 5/10 = \underline{6/12}$$

"The underlined names for 1/2 suggest something quite similar to the remarks about the broken eggs."

Activity 14. Cut squares about 2 × 2" out of cardboard that is white on one side, gray on the other. Or have the quares colored one color on one side, another color on the other. Put 12 squares in a box. Mix them up and dump them out on a table.

"How many squares are white? How many are gray? Without turning any over, arrange them in

a 3×4 array. Arrange them in different patterns." (Unless the result is 5 of one color and 7 of another, they can usually be arranged and rearranged to indicate many ways of saying the same thing. Suppose there are 4 white and 8 gray:



grey – 2 out of 3
white – 1 out of 3

4 out of 6
2 out of 6

8 out of 12
4 out of 12

After several examples, add 4 more squares so that there are 16 in all. Now all even numbers of squares, either white or gray (if one is even, so must the other be even), can be arranged in more than 1 way. The most useful results for making our point here are 8 – 8 and 12 – 4.

"The shorthand we have been using: 1/3, 2/5, 4/8, etc., can be used to keep track of our results. In the example when 8 out of 12 were gray, we can express the idea in the sketch this way:

Gray	White	All together
8/12 or 4/6 or 2/3	4/12 or 2/6 or 1/3	12/12, 6/6, 3/3, 11/11, 10/10, 9/9, etc.

"We can read this new shorthand: '8 out of 12 are gray' or '4 out of 6 are gray,' etc." Write the following on the board, and have pupils complete them.

If 8/12 are gray, then are white?
If 3/12 are white, then are gray?
If are white, then 7/12 are gray?

"Are the following statements true?"

If 8/12 are gray, then 1/3 is white?
If 1/4 is white, then 9/12 are gray?
If 2/12 are white, then 5/6 are gray?

Add 8 more squares. Now there are 24 in all. Proceed in same general way.

Talk about each example. Draw sketches. Encourage the children to explain their arguments. Keep asking, "Does that sound reasonable? Does everyone agree?"

Do not expect to reach any general rules about this idea that, later on, will be referred to as "ratio." We are introducing the idea at this point mainly for discussion purposes. We will come back to it.

Use the 2 × 2" squares of white-gray cardboard (or equivalent) for an early experience with the idea of chance or probability.

Put two squares in a box. Mix them, and drop them on a table. "How many are gray? In what different ways can squares show up?"

0 out of 2 may be gray
1 out of 2 may be gray
2 out of 2 may be gray

"Are there any other possible outcomes? Which result do you think will happen most often?"

Let several groups experiment, and have them each keep a record:

	How many are gray?		
	0/2	1/2	2/2
Totals	6	14	4

The more trials each group makes, the more the results will approximate a 1-2-1 ratio. That's a secret.

Then make a compilation of totals.

Group	0/2	1/2	2/2
A	6	14	4
B	4	10	8
C	7	15	4
D	8	13	8
Totals	25	52	24

Expand the activity to investigation of results when three squares are used. "What are the possible combinations? How many can be gray each time?"

0 out of 3
1 out of 3
2 out of 3
3 out of 3

Again, keep records.

	How many are gray?			
	0/3	1/3	2/3	3/3
Totals	5	14	16	6

Again, have several groups keep track of as many trials as possible—and then record the totals as

shown above. (This time, the outcomes will approximate 1-3-3-1—*that's a secret.*)

The only goals we have at this point are to plant the idea that certain combinations seem to occur more often than others; and to use familiar notations to express ratios. (For example, 2/3, meaning "2 out of 3.") However, we do not yet refer to the term *ratio.*

Activity 15. Experiment with notations and sketches such as follows:

"Here are two arrays of 15 squares. On one, we will mark one out of every three squares with an A. On the other, we will mark two out of every five squares with a B. How many squares out of the 15 will be marked in each case?"

A	A	A	A	A

1 out of 3,
or 5 out of 15

B	B			
B	B			
B	B			

2 out of every 5,
or 6 out of 15

Next show the same number of squares marked A and B together on a single sketch of 15 squares:

A	A	A	A	A
B	B			
B	B		B	B

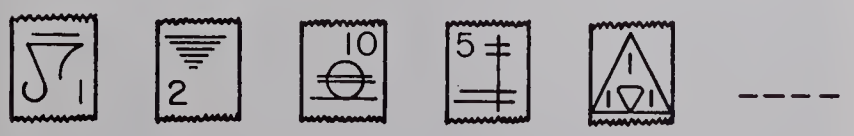
"If we were to simply put one diagram on top of the other, two squares would contain 2 marks. Since we want only one mark in any one square, we must shift two of the B's.

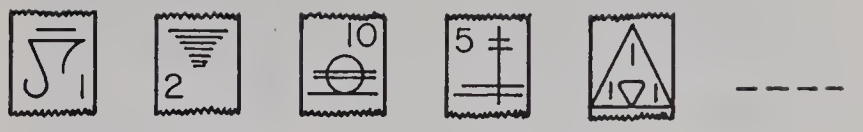
5 out of 15 and *6 out of 15* together are *11 out of 15*; or *1 out of 3* and *2 out of 5* together are *11 out of 15.*

"In shorthand, we might write:

5/15 and 6/15 → 11/15, or
1/3 and 2/5 → 11/15

Activity 16.
"Alice saved stamps. Her friend, Maxine, also saved stamps. Alice decided to give Maxine one out of each pair of duplicates. So, she arranged them in pairs across ther desk, something like this:





"How many did she give Maxine? Do you have enough information? Which of the following sentences need to be completed?

A. Alice has __ stamps arranged in pairs on her desk.
B. The stamps were worth $__.
C. She had __ pairs arranged on her desk.
D. Alice gave Maxine __ out of every pair of stamps.
E. Alice had __ stamps of which she had no duplicates.
F. Of those so arranged, she kept __ stamps.

Completions for any one of the sentences A, C, D, or F will provide enough information to answer the question conclusively.

Without completing A, C or F, have pupils draw a chart of some of the possibilities, if they know how many stamps Alice gave Maxine:

Number to Maxine	1	5	50		
No. of stamps arranged in pairs	2	10	100	etc.	

"In grocery stores, you will find cans, cartons, bottles and packages marked this way:

3/15	2/11	4/25
5/20	5/22	5/13

"What do you think those marks mean?" (3 for 15¢; 2 for 11¢; 4 for 25¢; etc.)

Activity 17. "If a can is marked 3/15, how much would you pay for 1 can?" Someone will say that if 3 cans cost 15¢, then 1 can would cost 5¢ because 15 ÷ 3 = 5. Someone else might contend that such is not the case, because then the cans would be marked 1/5. If the price is meant to indicate a bargain for purchasing 3 cans, then the price for 1 can must be 6¢. By purchasing 3 cans for 15¢ the customer saves 1¢ per can.

On the board draw a chart for each case:

No. of Cans	1	2	3	4	5	6	7	8	9	10
Price in cents	*5*	*10*	15			30				

No. of Cans	1	2	3	4	5	6	7	8	9	10
Price in cents	*6*	*12*	15	*21*	*27*	30				

"We agree that the price is 3 cans for 15c in either case. How much would 6 cans cost? How much would 9 cans cost? The same in both records?"

"Now, how about 1 can? We have agreed that 1 can might cost 5c or 6c. In each case, how much would 2 cans cost?" (10c in the first case; 12c in the second.)

But someone might argue that if 1 can costs 6c and 3 cans cost 15c, then you probably get a small bargain for buying 2 cans at a time. So, while 1 can cost 6c, 2 cans may cost only 11c. That would look like this:

		Savings
1 can	6 c	0
2 cans	11c	1c
3 cans	15c	3c

So, we need another chart:

No. of Cans	1	2	3	4	5	6	7
Price in cents	*6*	*11*	15			30	

Notes to accompany *Work Text* pages 56-57

Refer back to pages 48, 49 and 52 as introductions to pages 56 and 57. The same kind of illustrations that suggest the idea of multiplication of whole numbers will be employed to study multiplication of fractions.

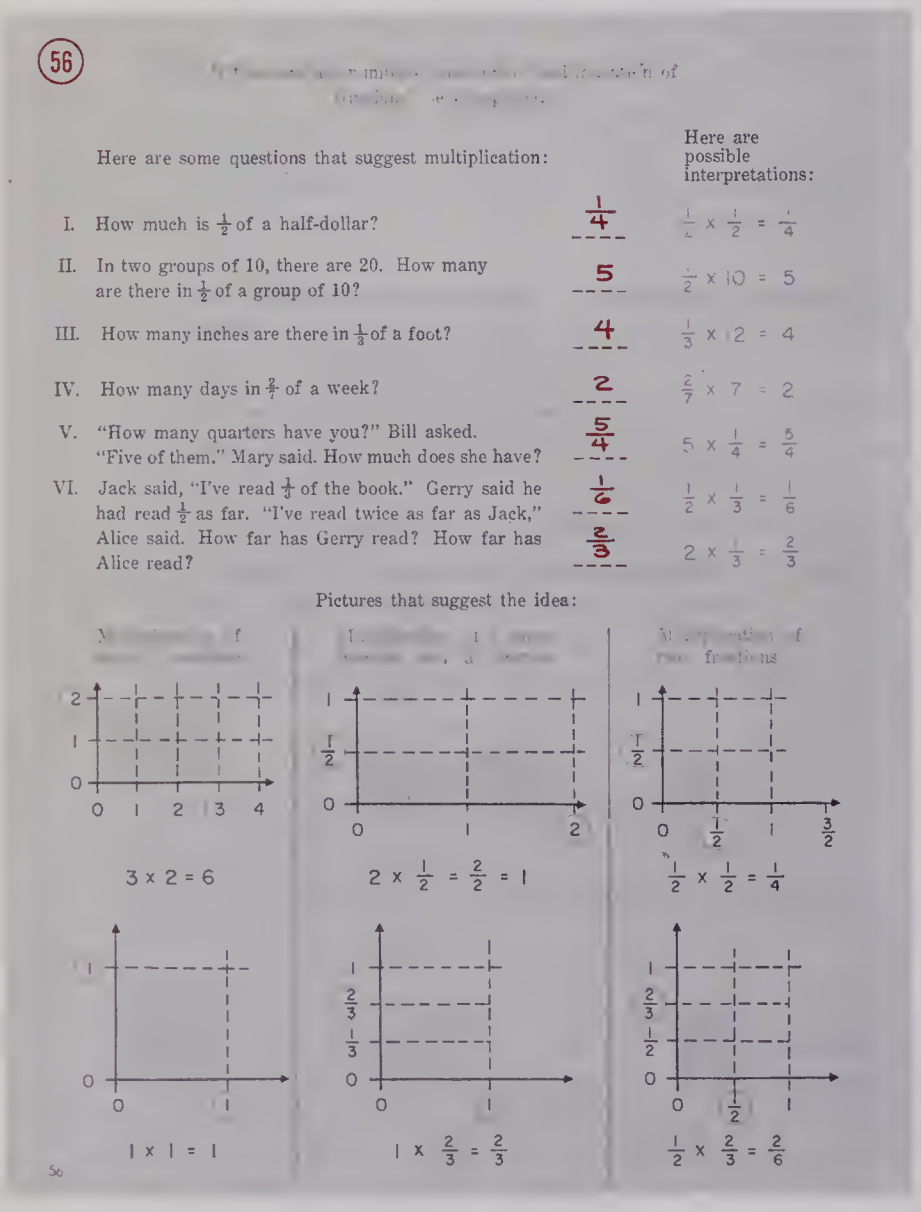
Many times, the idea of two groups of 5, for example, has been expressed mathematically as 2×5. Discuss the fact that "of" and "times" seem to be synonyms that express the idea implied by $\times$ (the multiplication sign). What we mean by "one-half times one-half" is quite abstract; what we mean by "one-half *of* one-half" is a description we use when that condition turns up in the real world.

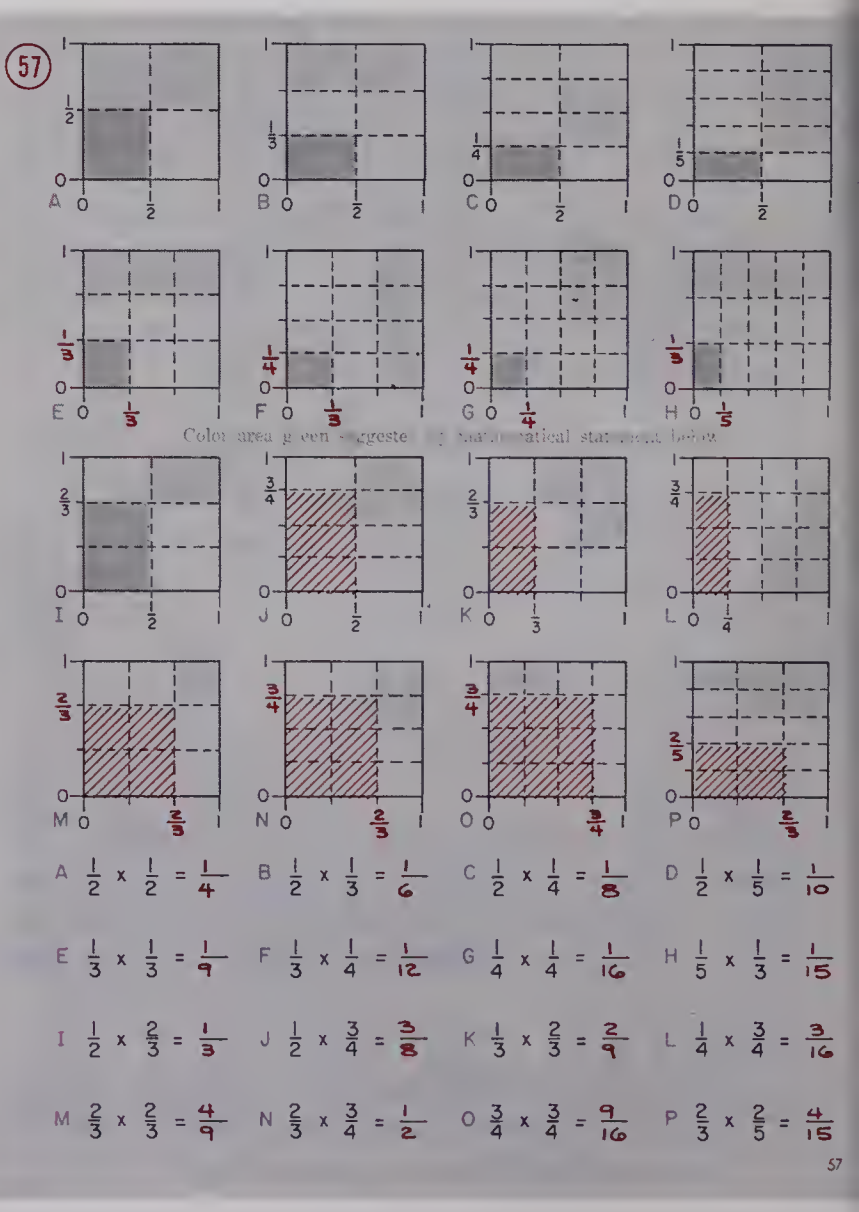
"Do you want half *of* the cake?"

"No, that's too much, but I would like a half *of* a half cake."

This is clearly a reference to one-fourth *of* a cake . . . or 1/2 of 1/2 = 1/4 and, synonymously, $1/2 \times 1/2 = 1/4$.

Work Text page 56 is designed to help fix the idea that "of" and "times" must have the same meaning.


56
Here are some questions that suggest multiplication:
Here are possible interpretations:
I. How much is 1/2 of a half-dollar? 1/4 — 1/2 x 1/2 = 1/4
II. In two groups of 10, there are 20. How many are there in 1/2 of a group of 10? 5 — 1/2 x 10 = 5
III. How many inches are there in 1/3 of a foot? 4 — 1/3 x 12 = 4
IV. How many days in 2/7 of a week? 2 — 2/7 x 7 = 2
V. "How many quarters have you?" Bill asked. "Five of them." Mary said. How much does she have? 5/4 — 5 x 1/4 = 5/4
VI. Jack said, "I've read 1/3 of the book." Gerry said he had read 1/2 as far. "I've read twice as far as Jack," Alice said. How far has Gerry read? How far has Alice read? 1/6 — 1/2 x 1/3 = 1/6; 2/3 — 2 x 1/3 = 2/3
Pictures that suggest the idea:
3 x 2 = 6
2 x 1/2 = 2/2 = 1
1/2 x 1/2 = 1/4
1 x 1 = 1
1 x 2/3 = 2/3
1/2 x 2/3 = 2/6
50



We are searching for a way to express mathematically the idea of "a half of a half-dollar." We consider $1/2 \times 1/2$ as a possibility. In a familiar type of sketch at bottom right on page 56, we are led to the conclusion that $1/2 \times 1/2 = 1/4$ . . . and this fits well with the fact that "a half of a half-dollar" is certainly "a quarter." Literally, perhaps, the sketch indicates that $1/2 \times 1/2$ is $1/4$ of 1×1 . . . or $1/2 \times 1/2 = 1/4 \times 1 = 1/4$

This exploratory approach is enlarged upon on page 57 to consider 16 different instances of regarding a pair of fractions as factors and looking for a reasonable candidate for "product." (Analagous representations on a number line would strengthen each conclusion.)

Notes to accompany *Work Text* pages 58-59

Greg Wilson takes what seems to be a quite dissimilar approach on pages 58 and 59 . . . and comes, surprisingly enough, to the same conclusions reached by using the sketches on pages 56 and 57 and by representing the ideas on a number line.

Greg is exploring a variation of the "row-column" or the "array" concept of multiplication (referred to by mathematicians as the "Cartesian product" or "cross product" definition of multiplication).

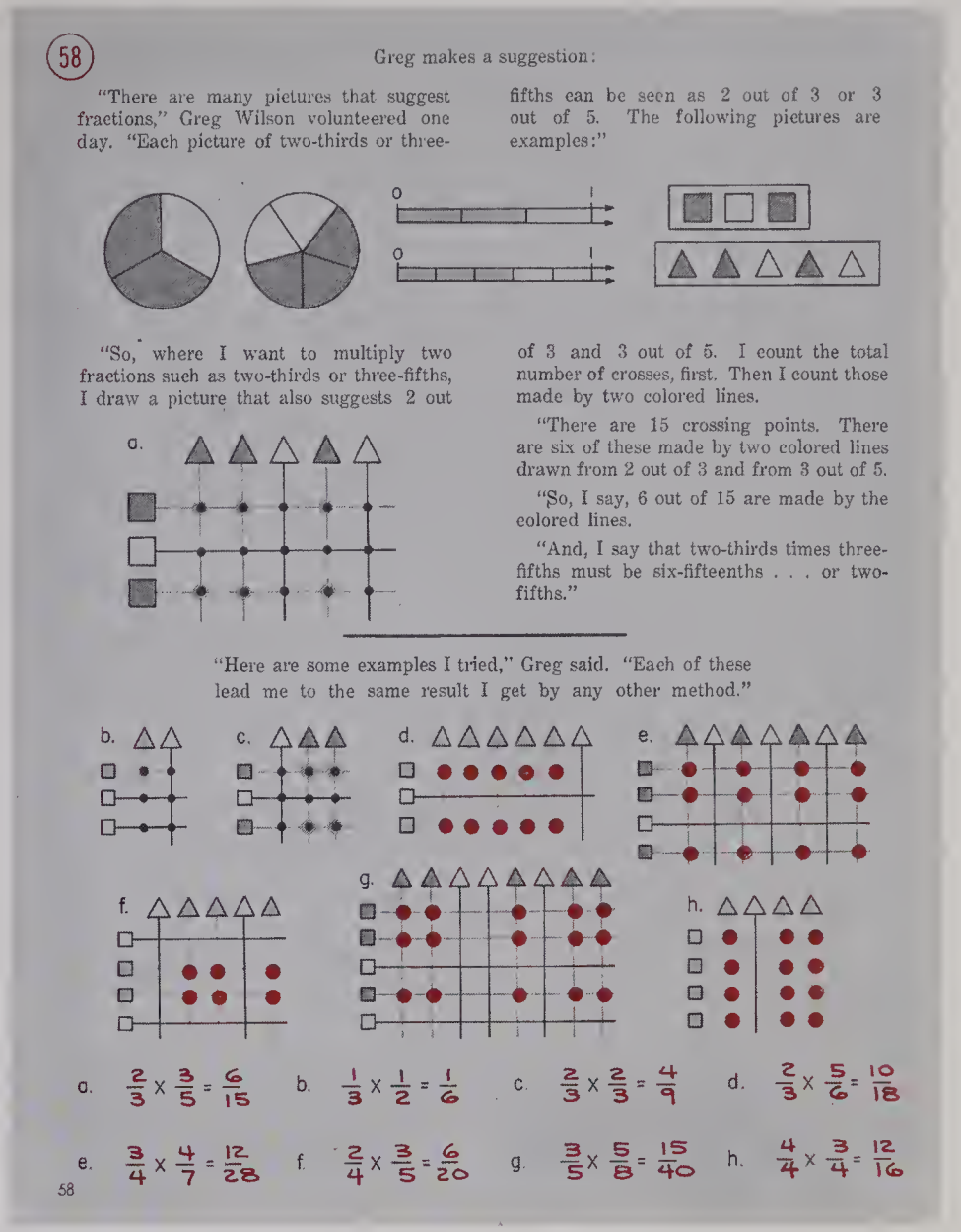
The important point here is that his unusual attack on the problem leads to the same results that have already been reached tentatively. We will not pursue this model or visual representation further, because it becomes a little unwieldy when we consider $1\ 1/3 \times 7/8 =$ ____. Our main interest is to find some sensible meaning for multiplication of fractions. The value in Greg's investigation is simply to convince us that we are on the right track.

Notes to accompany *Work Text* page 60

As children work through page 60, they will find a development that may not have been anticipated. If 1 is excluded, there is a unique set of prime factors for every whole number.

The request to express each number in terms of the "largest number of factors" is simply another way of saying that all factors must be "prime." The six can never occur as one of the largest number of factors because it can be expressed as 2×3, and thereby enlarge the group.

An agreement to write the prime factors in a certain order—smallest on the left to the largest on the right—has the result that all correct answers


58

Greg makes a suggestion:

"There are many pictures that suggest fractions," Greg Wilson volunteered one day. "Each picture of two-thirds or three-fifths can be seen as 2 out of 3 or 3 out of 5. The following pictures are examples:"

"So, where I want to multiply two fractions such as two-thirds or three-fifths, I draw a picture that also suggests 2 out of 3 and 3 out of 5. I count the total number of crosses, first. Then I count those made by two colored lines.

"There are 15 crossing points. There are six of these made by two colored lines drawn from 2 out of 3 and from 3 out of 5.

"So, I say, 6 out of 15 are made by the colored lines.

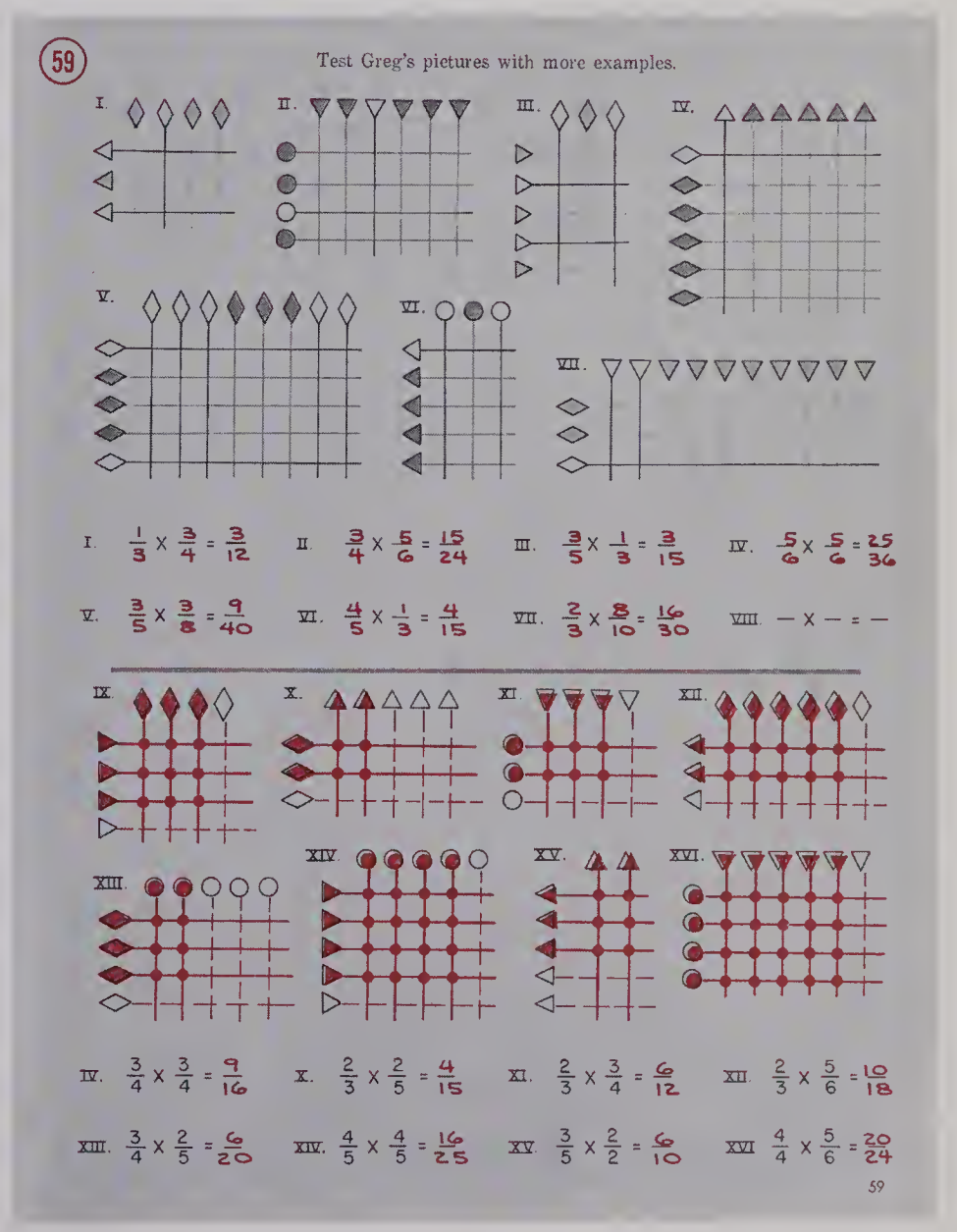
"And, I say that two-thirds times three-fifths must be six-fifteenths . . . or two-fifths."

"Here are some examples I tried," Greg said. "Each of these lead me to the same result I get by any other method."

a. $\frac{2}{3} \times \frac{3}{5} = \frac{6}{15}$ b. $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ c. $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ d. $\frac{2}{3} \times \frac{5}{6} = \frac{10}{18}$

e. $\frac{3}{4} \times \frac{4}{7} = \frac{12}{28}$ f. $\frac{2}{4} \times \frac{3}{5} = \frac{6}{20}$ g. $\frac{3}{5} \times \frac{5}{8} = \frac{15}{40}$ h. $\frac{4}{4} \times \frac{3}{4} = \frac{12}{16}$

58


59

Test Greg's pictures with more examples.

I. $\frac{1}{3} \times \frac{3}{4} = \frac{3}{12}$ II. $\frac{3}{4} \times \frac{5}{6} = \frac{15}{24}$ III. $\frac{3}{5} \times \frac{1}{3} = \frac{3}{15}$ IV. $\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

V. $\frac{3}{5} \times \frac{3}{8} = \frac{9}{40}$ VI. $\frac{4}{5} \times \frac{1}{3} = \frac{4}{15}$ VII. $\frac{2}{3} \times \frac{8}{10} = \frac{16}{30}$ VIII. $- \times - = -$

IV. $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$ X. $\frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$ XI. $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$ XII. $\frac{2}{3} \times \frac{5}{6} = \frac{10}{18}$

XIII. $\frac{3}{4} \times \frac{2}{5} = \frac{6}{20}$ XIV. $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$ XV. $\frac{3}{5} \times \frac{2}{2} = \frac{6}{10}$ XVI $\frac{4}{4} \times \frac{5}{6} = \frac{20}{24}$

59

(60)

Largest number of factors (other than 1)

3 and 4 are factors of 12. But 4 can be written as a pair of factors 2 and 2. So 12 can be written as a group of three factors: $12 = 2 \times 2 \times 3$. (Let us agree to write the smaller factors first, because order doesn't effect the results in multiplication.)

Product	Can it be factored?	Written as the most faetors	Written as other groups of faetors other than 1, with smallest factors listed first
12	yes	2 x 2 x 3	2 x 6 ; 3 x 4
8	yes	2×2×2	2×4
18	yes	2×3×3	2×9 ; 3×6
27	yes	3×3×3	3×9
24	yes	2×2×2×3	2×12; 3×8; 4×6; 2×3×4; 2×2×6
30	yes	2×3×5	5×6 ; 2×15 ; 3×10
20	yes	2×2×5	4×5 ; 2×10
23	no	—	—
35	yes	5×7	—
32	yes	2×2×2×2×2	2×2×2×4; 2×2×8; 2×16; 2×4×4; 4×8
36	yes	2×2×3×3	2×3×6; 2×2×9; 2×18; 3×12; 4×9
41	no	—	—
45	yes	3×3×5	3×15 ; 5×9
48	yes	2×2×2×2×3	2×2×2×6; 2×2×3×4; 2×2×12; etc.
49	yes	7×7	—
60	yes	2×2×3×5	2×2×15; 2×5×6; 4×15; 5×12; etc.
64	yes	2×2×2×2×2×2	8×8; 2×32; 4×16; 2×2×16; 4×4×4; etc.
72	yes	2×2×2×3×3	8×9; 4×18; 2×36; 3×24; 6×12; etc.
79	no	—	—

List all odd numbers less than 100 that cannot be factored if "1" is excluded

Addend	16	0	49	0	28	19	43	19	117	175	466
Addend	18	0	49	91	72	81	57	81	83	325	534
Sum	34	0	98	91	100	100	100	200	200	500	1000

(61) Jim Dooley said that the "largest number of factors" could be put to use in simplifying fractions. When asked to explain, he put the following on the blackboard:

$\frac{18}{27} = \frac{2 \times 3 \times 3}{3 \times 3 \times 3} \quad \frac{2}{3} \times \frac{9}{9} = \frac{2}{3} \times 1 = \frac{2}{3} \qquad \frac{20}{45} = \frac{2 \times 2 \times 5}{3 \times 3 \times 5} = \frac{4}{9} \times \frac{5}{5} = \frac{4}{9} \times 1 = \frac{4}{9}$

$\frac{30}{36} = \frac{2 \times 3 \times 5}{2 \times 2 \times 3 \times 3} = \frac{2 \times 3 \times 5}{2 \times 3 \times 2 \times 3} = \frac{6}{6} \times \frac{5}{6} = 1 \times \frac{5}{6} = \frac{5}{6}$

Let's test Jim Dooley's method.

	Written using largest number of factors	Rearrange faetors if necessary	Complete
$\frac{12}{18} =$	$\frac{2 \times 2 \times 3}{2 \times 3 \times 3} =$	$\frac{2 \times 3 \times 2}{2 \times 3 \times 3} =$	$\frac{6}{6} \times \frac{2}{3} = 1 \times \frac{2}{3} = \frac{2}{3}$
$\frac{9}{30} =$	$\frac{3 \times 3}{2 \times 3 \times 5} =$	$\frac{(3) \times 3}{(3) \times 2 \times 5} =$	$\frac{3}{3} \times \frac{3}{10} = 1 \times \frac{3}{10} = \frac{3}{10}$
$\frac{27}{45} =$	$\frac{3 \times 3 \times 3}{3 \times 3 \times 5} =$	$\frac{(3 \times 3) \times 3}{(3 \times 3) \times 5} =$	$\frac{9}{9} \times \frac{3}{5} = \frac{3}{5}$
$\frac{24}{36} =$	$\frac{2 \times 2 \times 2 \times 3}{2 \times 2 \times 3 \times 3} =$	$\frac{(2 \times 2 \times 3) \times 2}{(2 \times 2 \times 3) \times 3} =$	$\frac{12}{12} \times \frac{2}{3} = \frac{2}{3}$
$\frac{30}{20} =$	$\frac{2 \times 3 \times 5}{2 \times 2 \times 5} =$	$\frac{(2 \times 5) \times 3}{(2 \times 5) \times 2} =$	$\frac{10}{10} \times \frac{3}{2} = \frac{3}{2}$ (or $1 + \frac{1}{2}$ or $1\frac{1}{2}$)
$\frac{15}{60} =$	$\frac{3 \times 5}{2 \times 2 \times 3 \times 5} =$	$\frac{1}{2 \times 2}\frac{(3 \times 5)}{(3 \times 5)} =$	$\frac{1}{4} \times \frac{15}{15} = \frac{1}{4}$
$\frac{49}{35} =$	$\frac{7 \times 7}{5 \times 7} =$	$\frac{7}{5} \times \frac{(7)}{(7)} =$	$\frac{7}{5} \times \frac{7}{7} = \frac{7}{5}$ (or $1 + \frac{2}{5}$ or $1\frac{2}{5}$)
$\frac{17}{36} =$	$\frac{17}{2 \times 2 \times 3 \times 3} =$	$\frac{\quad}{\quad} =$	$\frac{17}{36}$
$\frac{54}{72} =$	$\frac{2 \times 3 \times 3 \times 3}{2 \times 2 \times 2 \times 3 \times 3} =$	$\frac{(2 \times 3 \times 3) \times 3}{(2 \times 3 \times 3) \times 2 \times 2} =$	$\frac{18}{18} \times \frac{3}{4} = \frac{3}{4}$

Jim has such an important idea, it has been given a name: "Reducing fractions to lowest terms."

Sum	4	0	13	22	27	101	16	17	18	25	110	
Addend	2	0	6	18	25	100	9	9	9	15	100	Factor
Addend	2	0	7	4	2	1	7	8	9	10	10	Factor
	4	0	42	72	50	100	63	72	81	150	1000	Product

in the center column will be the same! (The order in which the factors are written has no other function.)

"Make a list of all numbers that are listed in this central column." The list begins with 2, 3, 5, and 7 (since 1 is excluded). Others are 23, 41 and 79.

"What others might be included if there were more examples?" (Altogether, all prime numbers are candidates for this list.)

"Why do you think we agree to exclude 1 from our consideration of factors in this case?" The answer is simply that, otherwise, there would be an unlimited number of ways to express any number as a product of factors:

$$3 = 3 \times 1 = 1 \times 3 = 1 \times 1 \times 3 = 1 \times 3 \times 1$$
$$= 3 \times 1 \times 1 \times 1 \times 1 \ldots \text{ etc.}$$

When the column on the right has been completed, suggest to the children that they, "Draw a circle around each factor in this column of number names that is *not prime*. Did you include at least one factor in each number name in this column?" (Answer: Yes!)

"Could you write a name that belongs in this column and includes only prime numbers?" (No!)

"Why not?" Because it would simply repeat the group of factors listed in the center column . . . and the heading of the column on the right calls for "other groups."

(Some children may object to the exclusion of 1 as a prime factor. There should be no mystery about this. It is simply a matter of definition—or agreement. We have already indicated the purpose of this exclusion of 1 from our consideration . . . the alternative is to destroy the useful idea of "prime numbers" and "prime factors.")

Notes to accompany *Work Text* page 61

As the prime factor idea is put to use on page 61, one reason for the exclusion of 1 as a prime factor is clear. Its inclusion would create complete confusion and make further development of Jim Dooley's idea impossible.

The rule about writing the prime factors in order from smallest to largest may continue to apply in the column calling for the "largest number of factors," but must be relaxed in the next column on the right which calls for rearrangement in a more convenient order.

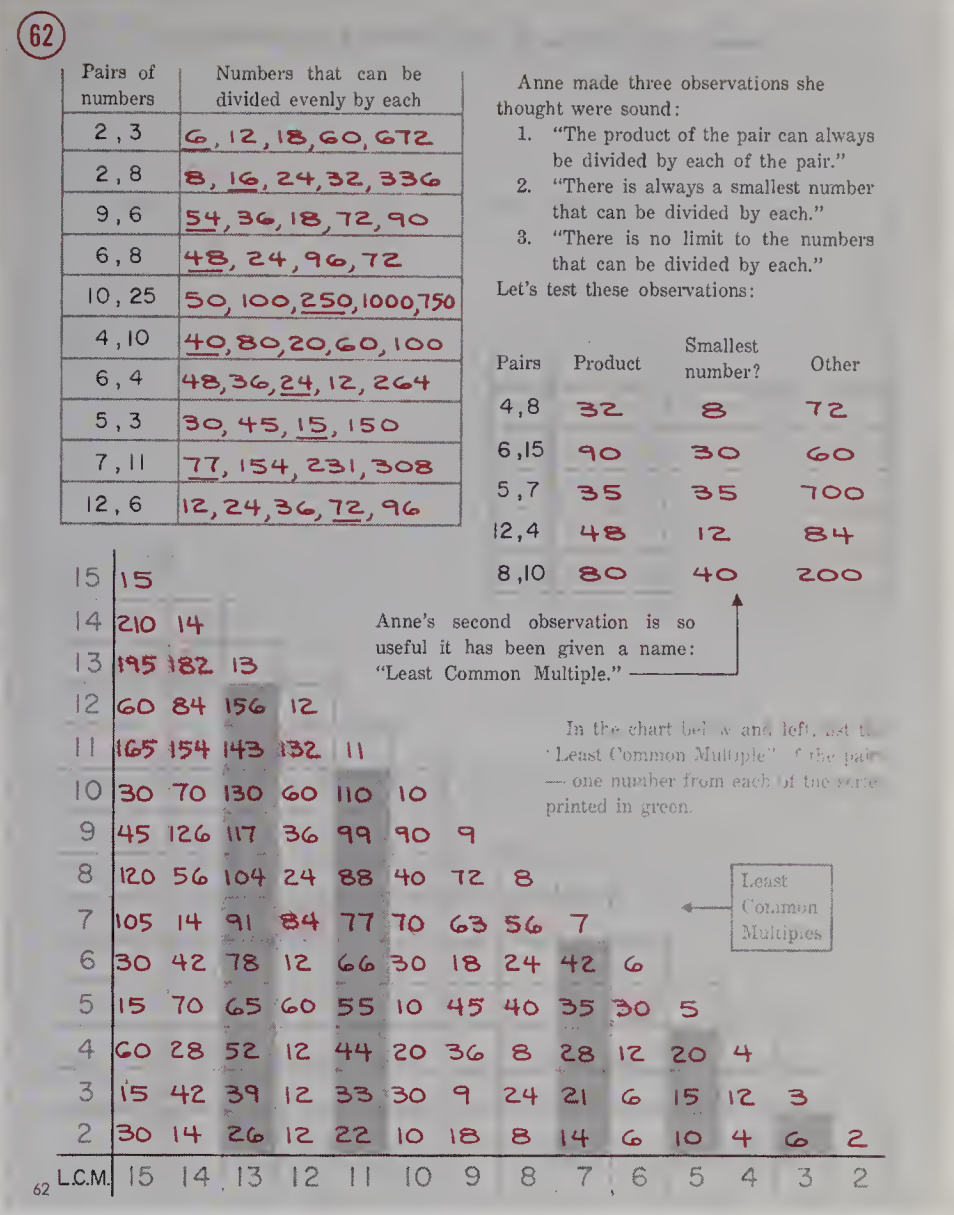
Many people refer to the results of Jim Dooley's idea as "reducing a fraction to lower terms." When the process has been carried to its conclusion and there is no common factor left in the numerator and denominator, people say that the fraction has been "reduced to lowest terms." You may want to familiarize your class with this language. However, our main attention is focused on the idea, not any particular name.

Again, complete mastery is not our goal. The idea will be developed again and again.

Notes to accompany *Work Text* page 62

On page 62, we are again concerned with factors and, particularly, with factors of a pair of numbers. (On page 63, we consider a special set of pairs—the denominators of a pair of fractions. The idea of "least common multiple" when employed in the situations studied on page 63, is often referred to as "finding the least common denominator," a special use of the more general idea explored on page 62.)

In the chart at the bottom of page 62, pairs are considered—one from the green numerals on the left, the other from the row of green numerals on the bottom. The "least common multiple" of the pair is noted in the appropriate cell.


62

Pairs of numbers	Numbers that can be divided evenly by each
2, 3	6, 12, 18, 60, 672
2, 8	8, 16, 24, 32, 336
9, 6	54, 36, 18, 72, 90
6, 8	48, 24, 96, 72
10, 25	50, 100, 250, 1000, 750
4, 10	40, 80, 20, 60, 100
6, 4	48, 36, 24, 12, 264
5, 3	30, 45, 15, 150
7, 11	77, 154, 231, 308
12, 6	12, 24, 36, 72, 96

Anne made three observations she thought were sound:

1. "The product of the pair can always be divided by each of the pair."
2. "There is always a smallest number that can be divided by each."
3. "There is no limit to the numbers that can be divided by each."

Let's test these observations:

Pairs	Product	Smallest number?	Other
4,8	32	8	72
6,15	90	30	60
5,7	35	35	700
12,4	48	12	84
8,10	80	40	200

Anne's second observation is so useful it has been given a name: "Least Common Multiple."

In the chart below and left, list the "Least Common Multiple" of the pairs — one number from each of the series printed in green.

15	15												
14	210	14											
13	195	182	13										
12	60	84	156	12									
11	165	154	143	132	11								
10	30	70	130	60	110	10							
9	45	126	117	36	99	90	9						
8	120	56	104	24	88	40	72	8					
7	105	14	91	84	77	70	63	56	7				
6	30	42	78	12	66	30	18	24	42	6			
5	15	70	65	60	55	10	45	40	35	30	5		
4	60	28	52	12	44	20	36	8	28	12	20	4	
3	15	42	39	12	33	30	9	24	21	6	15	12	3
2	30	14	26	12	22	10	18	8	14	6	10	4	6	2
L.C.M.	15	14	13	12	11	10	9	8	7	6	5	4	3	2

Least Common Multiples

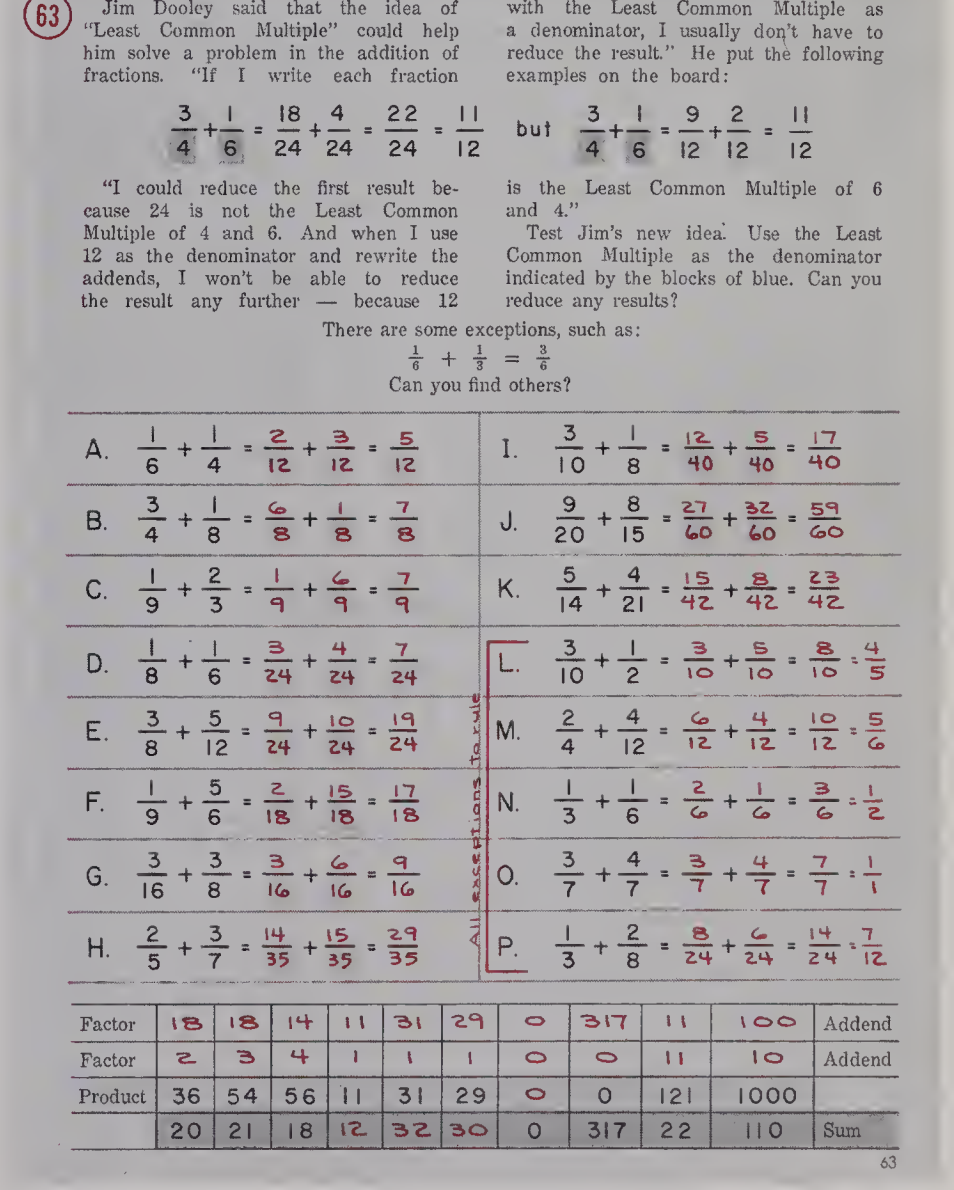
When this chart is completed, consider the numbers written in the green-tinted areas. They have a property in common: All can be expressed as the product of two primes, and those primes are noted in the left column and the bottom row. This property is unique for these numbers—there is no other number noted in the chart that has this property.

A second interesting group of entries are the products of the numbers noted by the green numerals in the particular row and column, but one or both are not prime numbers.

Finally, in the rest of the entries, the generating numbers have a common factor that was excluded in finding the "least common multiple."

Notes to accompany *Work Text* page 63

On page 63, Jim suggests a valuable idea—but he reaches an unwarranted conclusion. While we work through several examples of addition of fractions, using the least common multiple as denominator of each addend, let the discussion center on finding exceptions to Jim's rule. In this way, the activity is more than simply practice; it has a purpose that requires additional consideration.


63

Jim Dooley said that the idea of "Least Common Multiple" could help him solve a problem in the addition of fractions. "If I write each fraction with the Least Common Multiple as a denominator, I usually don't have to reduce the result." He put the following examples on the board:

$$\frac{3}{4}+\frac{1}{6}=\frac{18}{24}+\frac{4}{24}=\frac{22}{24}=\frac{11}{12} \quad \text{but} \quad \frac{3}{4}+\frac{1}{6}=\frac{9}{12}+\frac{2}{12}=\frac{11}{12}$$

"I could reduce the first result because 24 is not the Least Common Multiple of 4 and 6. And when I use 12 as the denominator and rewrite the addends, I won't be able to reduce the result any further — because 12 is the Least Common Multiple of 6 and 4."

Test Jim's new idea. Use the Least Common Multiple as the denominator indicated by the blocks of blue. Can you reduce any results?

There are some exceptions, such as:

$\frac{1}{6}+\frac{1}{3}=\frac{3}{6}$

Can you find others?

A.	$\frac{1}{6}+\frac{1}{4}=\frac{2}{12}+\frac{3}{12}=\frac{5}{12}$	I.	$\frac{3}{10}+\frac{1}{8}=\frac{12}{40}+\frac{5}{40}=\frac{17}{40}$
B.	$\frac{3}{4}+\frac{1}{8}=\frac{6}{8}+\frac{1}{8}=\frac{7}{8}$	J.	$\frac{9}{20}+\frac{8}{15}=\frac{27}{60}+\frac{32}{60}=\frac{59}{60}$
C.	$\frac{1}{9}+\frac{2}{3}=\frac{1}{9}+\frac{6}{9}=\frac{7}{9}$	K.	$\frac{5}{14}+\frac{4}{21}=\frac{15}{42}+\frac{8}{42}=\frac{23}{42}$
D.	$\frac{1}{8}+\frac{1}{6}=\frac{3}{24}+\frac{4}{24}=\frac{7}{24}$	L.	$\frac{3}{10}+\frac{1}{2}=\frac{3}{10}+\frac{5}{10}=\frac{8}{10}=\frac{4}{5}$
E.	$\frac{3}{8}+\frac{5}{12}=\frac{9}{24}+\frac{10}{24}=\frac{19}{24}$	M.	$\frac{2}{4}+\frac{4}{12}=\frac{6}{12}+\frac{4}{12}=\frac{10}{12}=\frac{5}{6}$
F.	$\frac{1}{9}+\frac{5}{6}=\frac{2}{18}+\frac{15}{18}=\frac{17}{18}$	N.	$\frac{1}{3}+\frac{1}{6}=\frac{2}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$
G.	$\frac{3}{16}+\frac{3}{8}=\frac{3}{16}+\frac{6}{16}=\frac{9}{16}$	O.	$\frac{3}{7}+\frac{4}{7}=\frac{3}{7}+\frac{4}{7}=\frac{7}{7}=\frac{1}{1}$
H.	$\frac{2}{5}+\frac{3}{7}=\frac{14}{35}+\frac{15}{35}=\frac{29}{35}$	P.	$\frac{1}{3}+\frac{2}{8}=\frac{8}{24}+\frac{6}{24}=\frac{14}{24}=\frac{7}{12}$

All exceptions to rule

Factor	18	18	14	11	31	29	0	317	11	100	Addend
Factor	2	3	4	1	1	1	0	0	11	10	Addend
Product	36	54	56	11	31	29	0	0	121	1000	
	20	21	18	12	32	30	0	317	22	110	Sum

63

As the exceptions are noted, several students—or the entire class—can be urged to search for more exceptions to Jim's rule—among other examples than those given on the page. Such exceptions are endless, of course.

"Can you find any pattern in the exceptions that would help you know at once whether an example will prove to be an exception to Jim's wrong conclusion?" This question has no easy answer. There is no simple rule that can be found at this time. Of course, if there is a common factor in both numerator and denominator (such as "2" in example M), it will be an exception. Children may find other clues—some reliable and some unreliable. Have each candidate adequately tested by the class.

Notes to accompany *Work Text* page 64

Pages 64, 65 and 67 are based on the important idea that "numbers have many names" . . . often referred to in many cases as "regrouping."

Along with the work on page 64, which considers several different ways to name each of the points indicated on number lines, you should discuss similar patterns in the real world.

"Is one dollar and a half the same amount of money as three half-dollars and one and one-half dollars?"

"Is two quarters the same amount of money as a half-dollar? Are two half-dollars the same amount as 4 quarters?"

"In how many ways can you make 2 dollars with quarters and halves?"

No. of half-dollars	No. of quarters	A similar pattern in arithmetic
4	0	4/2 + 0/4 = 2
3	2	3/2 + 2/4 = 2
2	4	2/2 + 4/4 = 2
1	6	1/2 + 6/4 = 2
0	8	0/2 + 8/4 = 2

After a chart such as the one above left is completed, indicate after each instance the expression from arithmetic that names each pattern.

Similar activities can be developed around other real life situations. For example: Is one gallon the same amount of liquid as two half-gallons? . . .the same as four quarter-gallons (quarts)? . . .the same as eight pints? . . .the same as 16 half-pints (glass)? With this understanding, you can play the game of "fewest cartons." If milk is available in cartons of each of the measures mentioned above, how can you get a specified amount of milk in the fewest cartons?

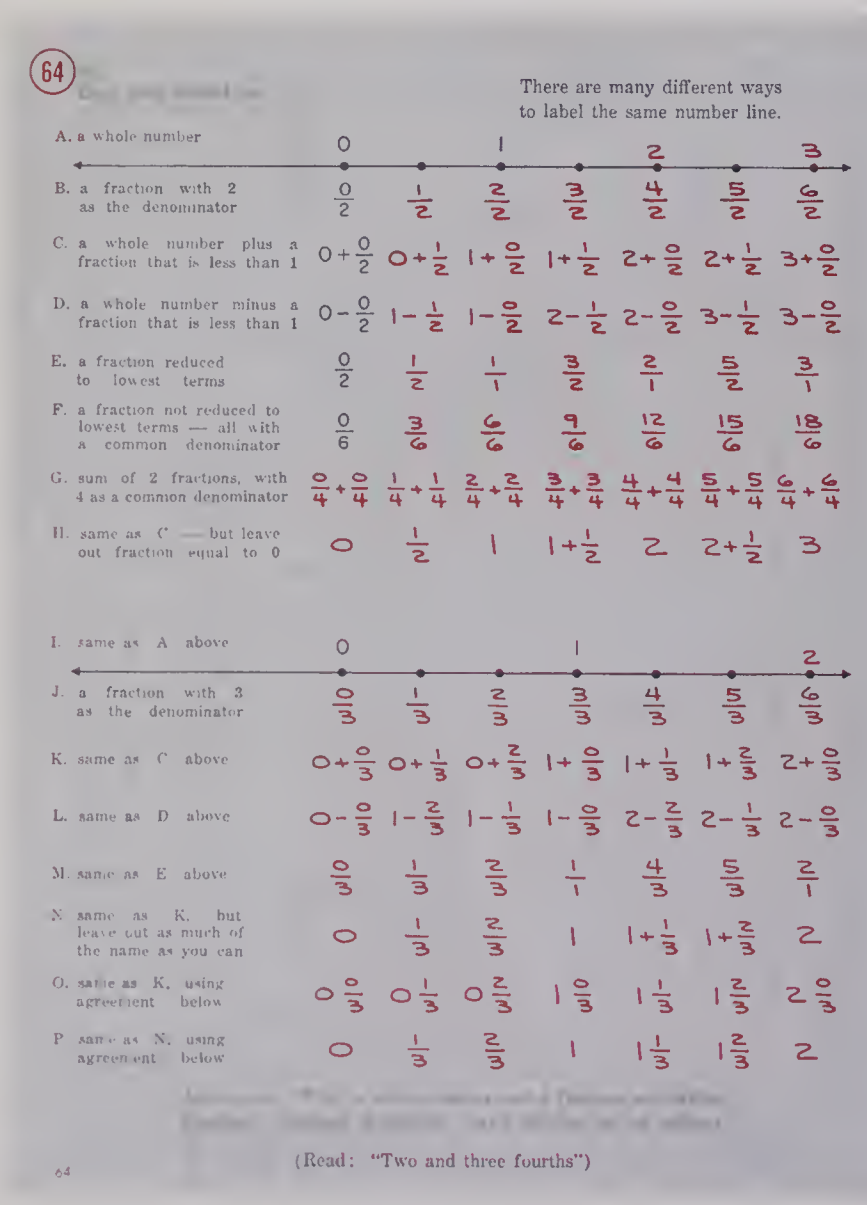
Amount to be purchased	No. of gals.	No. of half-gals.	No. of quarts	No. of pints	No. of half-pints
5 quarts	1	0	1	0	0
3 half-gallons	1	1	0	0	0
3 half-pints	0	0	0	1	1
11 half-pints	0	1	0	1	1

The game can be shifted as an example of "In How Many Ways?" . . .

With the cartons mentioned above, in how many ways could you purchase the specified amounts of milk?

Amount	G	HG	Q	P	HP
1 quart	0	0	1	0	0
	0	0	0	1	2
	0	0	0	2	0
	0	0	0	0	4
2 quarts	0	0	2	0	0

Such activities point to patterns in our systems of measurement in which a given amount of distance can be expressed in many ways—paralleling the idea that "numbers have many names."


64
There are many different ways to label the same number line.
A. a whole number
B. a fraction with 2 as the denominator
C. a whole number plus a fraction that is less than 1
D. a whole number minus a fraction that is less than 1
E. a fraction reduced to lowest terms
F. a fraction not reduced to lowest terms — all with a common denominator
G. sum of 2 fractions, with 4 as a common denominator
H. same as C — but leave out fraction equal to 0
I. same as A above
J. a fraction with 3 as the denominator
K. same as C above
L. same as D above
M. same as E above
N. same as K, but leave out as much of the name as you can
O. same as K, using agreement below
P. same as N, using agreement below
(Read: "Two and three fourths")
64

Notes to accompany *Work Text* page 65

Page 65 introduces the idea of "Mixed Number." All points on the number line we have encountered thus far can be names by expressions in the form of a sum of a whole number (including 0) and a fraction (including fractions that have 0 as the numerator—excluding 0 as a denominator because such a fraction would indicate division by 0, a concept banned from arithmetic because we can't find a reasonable meaning for such an expression).

This is an extension of the idea of "regrouping" or that "numbers have many names." An understanding of the wide freedom to rename mixed numbers is essential to addition, subtraction, multiplication and division of mixed numbers. There is no effort to deal with each operation as a type, since such separation would only lead away from understanding and end up with a set of "rules."

Before the page is far underway, the student encounters 2 1/2 − 3/4 and 4 1/4 − 1 1/2. Rather than becoming involved in the business of "least common denominators" and "borrowing," let children find a sensible answer in any way they want, such as drawing out numbers on a number line.

As often happens, a discussion of examples involving money will help broaden the understanding of what's going on.

"If you had 2 1/2 dollars and spent 3 quarters (of a dollar), how much would you have left?" A mathematical expression that reflects this idea is: 2 1/2 − 3/4 = 1 3/4.

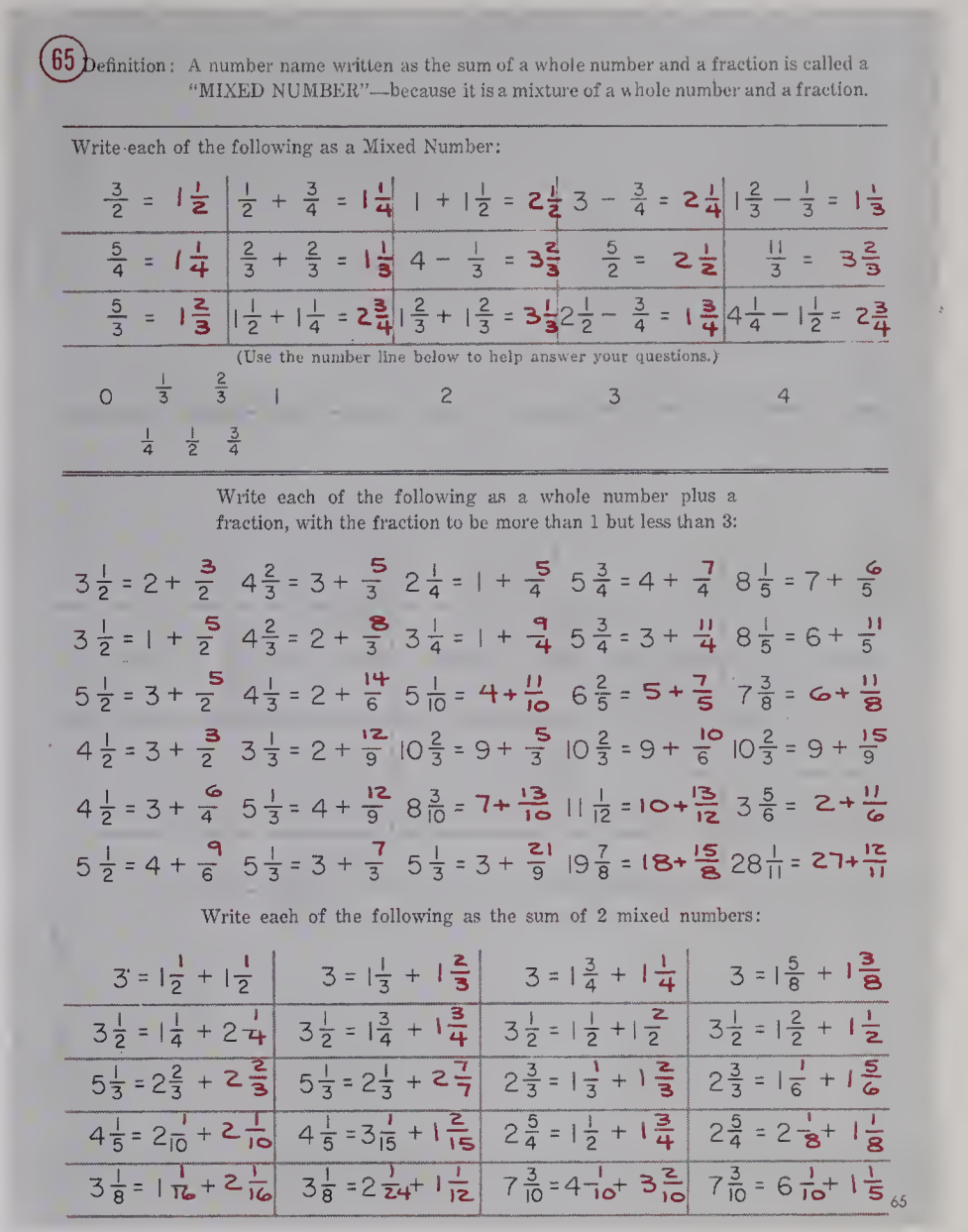
"A balloon went up 1 2/3 miles in the air and was then pulled back down. That's a round trip of 1 2/3 miles each way. How far up and down did it go?"

1 2/3 + 1 2/3 = 3 1/3 or 2 × 1 2/3 = 3 1/3.
Answer: 3 1/3 miles.

"Each of two sticks is 1 2/3 yards long. Placed end to end, they wouldn't read 4 yards. They would be how much short of 4 yards?"

1 2/3 + 1 2/3 = 4 − 2/3 = 3 + 1/3, or 3 1/3

Whenever an example presents a difficulty, use the number line, examples from the physical world and other diagrams as a means to help reach a reasonable conclusion. Most children, if given time to think through many examples, will find their own shortcuts and will never have to be supplied with an algorism or procedure. By analogy with earlier work in fractions, they will need no crutches—provided they understand what a mixed number is and have some experience in some of the many ways in which other names can be found. Thus equipped, they will never need to be given any form or series of steps to follow in handling addition and subtraction of mixed numbers.



65 Definition: A number name written as the sum of a whole number and a fraction is called a "MIXED NUMBER"—because it is a mixture of a whole number and a fraction.

Write each of the following as a Mixed Number:

$\frac{3}{2} = 1\frac{1}{2}$	$\frac{1}{2} + \frac{3}{4} = 1\frac{1}{4}$	$1 + 1\frac{1}{2} = 2\frac{1}{2}$	$3 - \frac{3}{4} = 2\frac{1}{4}$	$1\frac{2}{3} - \frac{1}{3} = 1\frac{1}{3}$
$\frac{5}{4} = 1\frac{1}{4}$	$\frac{2}{3} + \frac{2}{3} = 1\frac{1}{3}$	$4 - \frac{1}{3} = 3\frac{2}{3}$	$\frac{5}{2} = 2\frac{1}{2}$	$\frac{11}{3} = 3\frac{2}{3}$
$\frac{5}{3} = 1\frac{2}{3}$	$1\frac{1}{2} + 1\frac{1}{4} = 2\frac{3}{4}$	$1\frac{2}{3} + 1\frac{2}{3} = 3\frac{1}{3}$	$2\frac{1}{2} - \frac{3}{4} = 1\frac{3}{4}$	$4\frac{1}{4} - 1\frac{1}{2} = 2\frac{3}{4}$

(Use the number line below to help answer your questions.)

0 $\frac{1}{3}$ $\frac{2}{3}$ 1 2 3 4
$\frac{1}{4}$ $\frac{1}{2}$ $\frac{3}{4}$

Write each of the following as a whole number plus a fraction, with the fraction to be more than 1 but less than 3:

$3\frac{1}{2} = 2 + \frac{3}{2}$ $4\frac{2}{3} = 3 + \frac{5}{3}$ $2\frac{1}{4} = 1 + \frac{5}{4}$ $5\frac{3}{4} = 4 + \frac{7}{4}$ $8\frac{1}{5} = 7 + \frac{6}{5}$

$3\frac{1}{2} = 1 + \frac{5}{2}$ $4\frac{2}{3} = 2 + \frac{8}{3}$ $3\frac{1}{4} = 1 + \frac{9}{4}$ $5\frac{3}{4} = 3 + \frac{11}{4}$ $8\frac{1}{5} = 6 + \frac{11}{5}$

$5\frac{1}{2} = 3 + \frac{5}{2}$ $4\frac{1}{3} = 2 + \frac{14}{6}$ $5\frac{1}{10} = 4 + \frac{11}{10}$ $6\frac{2}{5} = 5 + \frac{7}{5}$ $7\frac{3}{8} = 6 + \frac{11}{8}$

$4\frac{1}{2} = 3 + \frac{3}{2}$ $3\frac{1}{3} = 2 + \frac{12}{9}$ $10\frac{2}{3} = 9 + \frac{5}{3}$ $10\frac{2}{3} = 9 + \frac{10}{6}$ $10\frac{2}{3} = 9 + \frac{15}{9}$

$4\frac{1}{2} = 3 + \frac{6}{4}$ $5\frac{1}{3} = 4 + \frac{12}{9}$ $8\frac{3}{10} = 7 + \frac{13}{10}$ $11\frac{1}{12} = 10 + \frac{13}{12}$ $3\frac{5}{6} = 2 + \frac{11}{6}$

$5\frac{1}{2} = 4 + \frac{9}{6}$ $5\frac{1}{3} = 3 + \frac{7}{3}$ $5\frac{1}{3} = 3 + \frac{21}{9}$ $19\frac{7}{8} = 18 + \frac{15}{8}$ $28\frac{1}{11} = 27 + \frac{12}{11}$

Write each of the following as the sum of 2 mixed numbers:

$3 = 1\frac{1}{2} + 1\frac{1}{2}$	$3 = 1\frac{1}{3} + 1\frac{2}{3}$	$3 = 1\frac{3}{4} + 1\frac{1}{4}$	$3 = 1\frac{5}{8} + 1\frac{3}{8}$
$3\frac{1}{2} = 1\frac{1}{4} + 2\frac{1}{4}$	$3\frac{1}{2} = 1\frac{3}{4} + 1\frac{3}{4}$	$3\frac{1}{2} = 1\frac{1}{2} + 1\frac{2}{2}$	$3\frac{1}{2} = 1\frac{2}{2} + 1\frac{1}{2}$
$5\frac{1}{3} = 2\frac{2}{3} + 2\frac{2}{3}$	$5\frac{1}{3} = 2\frac{1}{3} + 2\frac{7}{7}$	$2\frac{3}{3} = 1\frac{1}{3} + 1\frac{2}{3}$	$2\frac{3}{3} = 1\frac{1}{6} + 1\frac{5}{6}$
$4\frac{1}{5} = 2\frac{1}{10} + 2\frac{1}{10}$	$4\frac{1}{5} = 3\frac{1}{15} + 1\frac{2}{15}$	$2\frac{5}{4} = 1\frac{1}{2} + 1\frac{3}{4}$	$2\frac{5}{4} = 2\frac{1}{8} + 1\frac{1}{8}$
$3\frac{1}{8} = 1\frac{1}{16} + 2\frac{1}{16}$	$3\frac{1}{8} = 2\frac{1}{24} + 1\frac{1}{12}$	$7\frac{3}{10} = 4\frac{1}{10} + 3\frac{2}{10}$	$7\frac{3}{10} = 6\frac{1}{10} + 1\frac{1}{5}$

65

Notes to accompany *Work Text* page 66

In handling mixed numbers as factors, you may want to refer to Ben's version of "Do-It-to-Both."

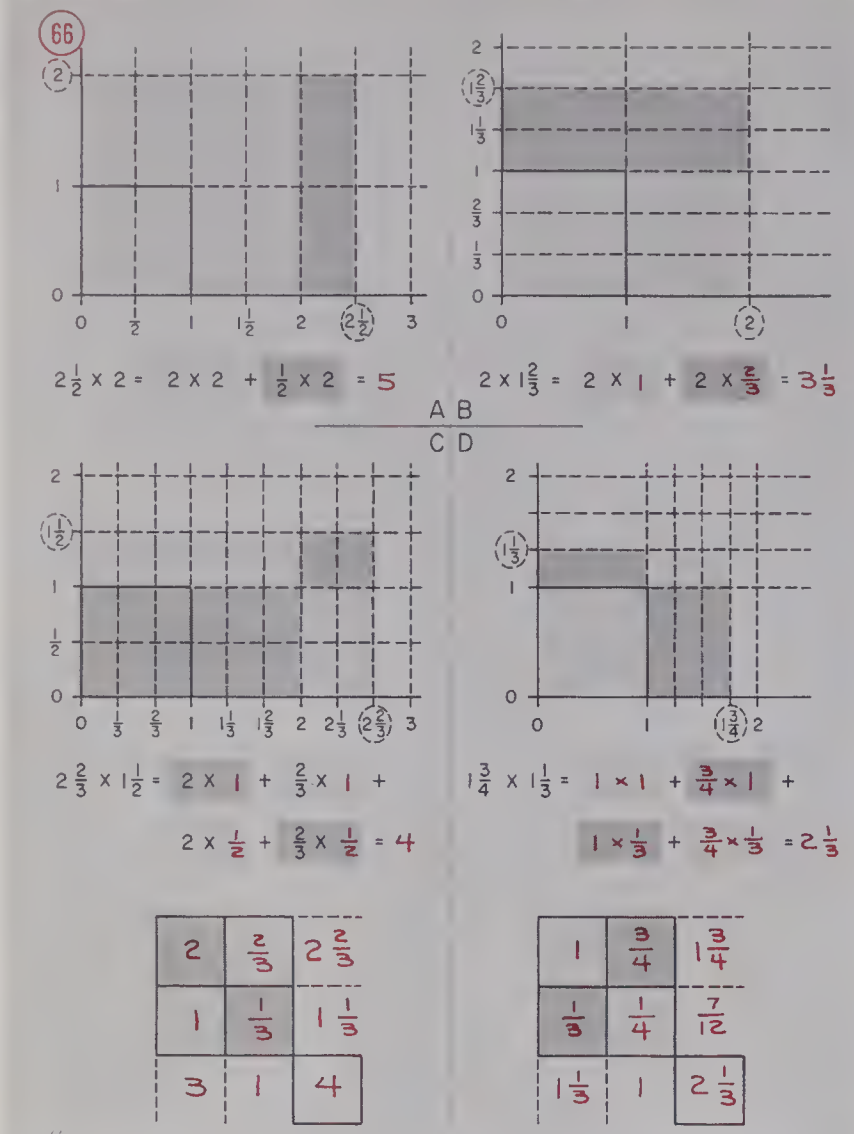
2 1/2 × 2 = (2+1/2) × 2 = 2 · 2 + 1/2 · 2
= 4 + 1 = 5
2 × 1 2/3 = 2(1+2/3) = 2 · 1 + 2 · 2/3
= 2 + 4/3 = 2 + 1 + 1/3 = 3 1/3

Examples C and D are "Double-do-it-to-both":

$2\frac{2}{3} \times 1\frac{1}{2} = (2 + \frac{2}{3}) \times (1 + \frac{1}{2})$

$= 2 \cdot 1 + 2 \cdot \frac{1}{2} + \frac{2}{3} \cdot 1 + \frac{2}{3} \cdot \frac{1}{2}$

$= 2 + 1 + \frac{2}{3} + \frac{1}{3} = 4$


66
$2\frac{1}{2} \times 2 = 2 \times 2 + \frac{1}{2} \times 2 = 5$
$2 \times 1\frac{2}{3} = 2 \times 1 + 2 \times \frac{2}{3} = 3\frac{1}{3}$
A B
C D
$2\frac{2}{3} \times 1\frac{1}{2} = 2 \times 1 + \frac{2}{3} \times 1 +$
$2 \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} = 4$
$1\frac{3}{4} \times 1\frac{1}{3} = 1 \times 1 + \frac{3}{4} \times 1 +$
$1 \times \frac{1}{3} + \frac{3}{4} \times \frac{1}{3} = 2\frac{1}{3}$
2 | 2/3 | 2 2/3
1 | 1/3 | 1 1/3
3 | 1 | 4
1 | 3/4 | 1 3/4
1/3 | 1/4 | 7/12
1 1/3 | 1 | 2 1/3

66

A Cross Number Puzzle form is added in the second pair of examples to help keep track of parts of the problem. As usual, develop discussion of this pattern as it is found in real life.

"I earned 2 1/2 dollars on each of two days. My total earnings were 5 dollars" . . . 2 1/2 × 2 = 5.

"There are 100 minutes in 1 2/3 hours. Twice that amount of time is 200 minutes, or 3 hours and 20 minutes . . . or 3 1/3."

"One ladder was 2 2/3 yards long and another was half as long. Together, they would reach 8 feet plus 4 feet, or 12 feet . . . which is 4 yards."

"A man worked 1 1/3 hours at $1.75 per hour. For one hour, he received $1.75 and 1/3 of that for the rest of the time—or $.58 1/3. That would be $2.33 1/3 . . . or $2 1/3."

"In a package of cookies there were three rows with seven cookies in a row. That is 1 3/4 dozen. I brought one full box and a box that had only 1/3 of the cookies left—one row of seven. So, I bought 21 plus 7, or 28 cookies in all. That is 2 4/12 dozen, or 2 1/3 dozen."

Analyzing the sketches

When considering the sketches on page 66, analyze each one. In example A, the whole sketch suggests the idea of 2 × 2 1/2 or 2 1/2 × 2. If that is agreed, then what does the blue shaded area suggest? The idea of 2 × 2, and it is four times as large as the area inside the heavy black line that suggest 1 × 1 or 1.

What does the green area suggest? The idea of 1/2 × 2 or 2 × 1/2. Each of the small sections suggest 1/2 of 1 or 1/2. In the green area, there are two such sections, and 2 × 1/2 = 1. The green area could be cut into two chunks that would fill the area inside the heavy black line. If the blue area suggests 4 and the green area 1, the shaded areas together suggest 5.

In example C, the sketch suggests the idea of 1 1/2 × 2 2/3. Four areas are shaded. The large green area suggests that 1 × 2 = 2 × 1 = 2. The blue area above suggests 1/2 × 2 = 2 × 1/2 = 2/2 = 1.

Since each small section seems to be 1/6 of the area inside the solid black line (1×1 = 1), the blue area above contains six of those small regions, and 6 × 1/6 = 1. Certainly a reasonable result.

The other blue area contains four of the small regions, and 4 × 1/6 = 4/6. The area suggests the idea of 1 × 2/3 or 2/3 . . . and we note that 4/6 = 2/3.

The small green area suggests the idea of 1/2 × 2/3 or 2/6. We count two small regions, and 2 × 1/6 = 2/6 or 1/3.

Putting the areas together is suggested by 2 + 1 + 2/3 + 1/3 = 2 + 1 + 1 = 4.

Approaching each example from several directions and reaching a reasonable conclusion without resorting to any rules helps to build a basic understanding and confidence in children. They are prepared for the following summary:

$$\begin{array}{lll} & \downarrow & \\ 2\ 1/2 \times 2 = 5/2 \times 2/1 = 10/2 & & = 5 \\ 2 \times 1\ 2/3 = 2/1 \times 5/3 = 10/3 & & = 3\ 1/3 \\ 2\ 2/3 \times 1\ 1/2 = 8/3 \times 3/2 = 24/6 & & = 4 \\ 1\ 3/4 \times 1\ 1/3 = 7/4 \times 4/3 = 28/12 & & = 2\ 1/3 \end{array}$$

Notice the expressions written under the arrow. In the first example, the picture contains 10 small regions and each suggests one half or a total of 10 halves. The sketch of B contains 10 small regions, each representing 1/3. The sketch of C contains 24 small regions, each representing 1/6. In the sketch of D, were the grid of broken lines completed there

would be 28 regions, each suggesting the idea of 1/12.

Notes to accompany *Work Text* page 67

Suppose the chart on this page were unlimited—as indicated by the arrows. Then it would include any fraction with whole numbers as numerator and denominator. To find "14/17" we would go out to the left on the scale at the bottom to 14; then move up that column until we reach the row opposite the blue numeral 17 on the left . . . and there it is.

If the chart were extended we could find, say, 1,297/91,275, by using the same method—right to 1,297 and up to 91,275. Indeed, we can see some patterns that would extend on out into our endlessly expanded imaginary chart. Consider the fractions noted in the solid blue cells. The series is 1/2, 2/3, 3/6, 4/8, etc., and we recognize them all as other names for 1/2. "How does one get from one of these names to the next one further out on the chart?" Answer: Go over 1 and up 2. If we wanted to skip every other name in this group, we would go over 2 and up 4, etc.

Next, the series in the solid green cells begins 1/3, 2/3, 3/9, 4/12, etc. All are other names for 1/3. And to move from one name to the next one further out on the chart, we move over to the right 1 and up 3. To skip every other name, we move over 2 and up 6.

"Color in the various names of 1/4. Do they seem to be arranged so you could connect them all with a line drawn with a ruler?" Yes.

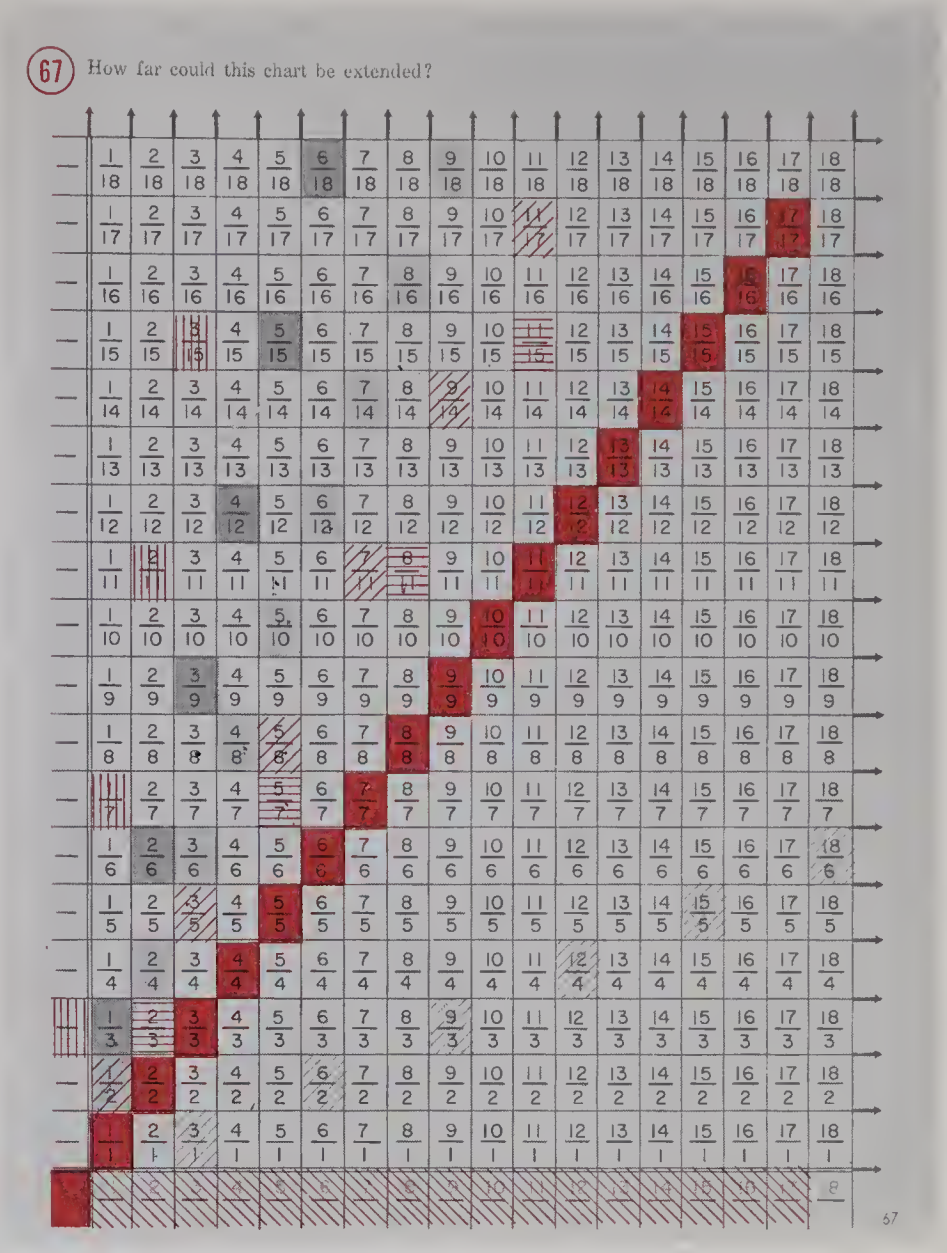
Similarly, consider the series of names shaded with the blue line, the series shaded with the green lines . . . and any other groups of names for the same fractions. "Can you find a series of names for 1?" Yes: 1/1, 2/2, 3/3, etc.

"Can you find a series of names for the whole numbers in which the numerator and denominator have no common factor?" Yes: 1/1, 2/1, 3/1, 4/1, etc., the bottom row just above the green numerals.

More patterns

Have the children color in the series of other names for 1. "In what way do all the fractions written above this diagonal differ from all those written below the diagonal?" Answer: Above, they are all less than 1 and below, they are all greater than 1.

"Where would you look for other names for 'mixed numbers?'" Answer: Below the diagonal. "Find all the names given for 1 1/2." Answer 3/2, 6/4, 9/6, 12/8, etc. Again, they could all be connected by a line drawn with a ruler.



Later the children will work with "unit fractions" and these should be pointed out. The left-hand row is composed of fractions whose numerator is 1. For centuries, the early Egyptian and Arabic mathematicians wrote all fractions as the sum of unit fractions. They would write 3/4 as 1/2 + 1/4; 11/12 would be written as 1/2 + 1/3 + 1/12 or 1/2 + 1/4 + 1/6, etc.

Patterns in fractions abound when they are organized as they are on this page. The discovery of such patterns helps to dispel the idea that fractions are a confusing collection of unrelated names for numbers.

Without laboring the point, mention that when men invented fractions so we would have an answer to 7 ÷ 9, 5 ÷ 2, etc., he did so in such a way as to include the whole numbers which had become inadequate—the bottom row is eloquent testimony that the old was included in the new.

Notes to accompany *Work Text* page 68

Pages 68 through 77 consider another invention by men who desired a system in which subtraction was always possible. When the ancients ran up against

THE COUNTDOWN.

"Blast off minus one minute," came over the loud speaker.

"Blast off minus 30 seconds."

Bill looked out of the window and thought, "Well, Mr. Moon, here we come!"

"Blast off minus 10 seconds . . . minus 5 seconds . . . minus 4, minus 3, minus 2, minus 1 . . . Blast off!"

"Blast off plus 1 . . . plus 2 . . . plus 3 . . ."

As the count went on, Bill looked out again to see the earth falling away below.

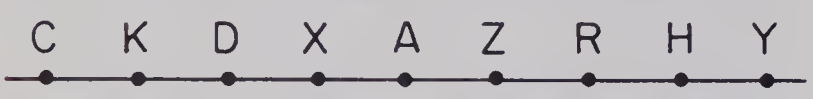
When the announcer reached ". . . plus 30 seconds . . ." he stopped counting and shouted happily: "Everything is A. O. K."

How long is it between two calls?

1st Call	2nd Call	Seconds Between
minus 5	minus 3	2
minus 7	BLAST OFF	7
minus 6	minus 1	5
BLAST OFF	plus 4	4
minus 1	plus 1	2
minus 5	plus 5	10
minus 3	plus 4	7
minus 8	minus 2	6

the expression 2 – 3, they dismissed it as meaningless. We do not, of course.

"Directions" make a great deal of difference in many events that we recognize in the real world. You can illustrate this idea by drawing a number line on the board, with points labeled at random with letters.



"Let's suppose this suggests a road with sign posts a mile apart, and labeled as indicated. Robert started from R and went 2 miles. Dick started at D and went 2 miles. How far were they apart at that time?"

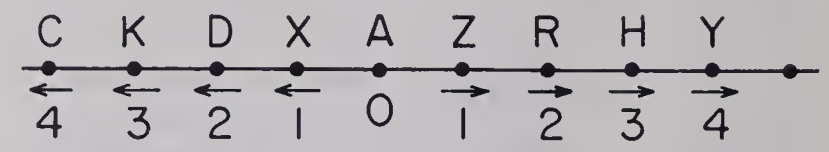
There are four possible answers:
If Dick went to X and Robert went to A, they are 1 mile apart.
If Dick went to X and Robert went to Y, they are 5 miles apart.
If Dick went to K and Robert went to A, they are 3 miles apart.
If Dick went to K and Robert went to Y, they are 7 miles apart.

Suppose Alice started at A. She walked 2 miles before lunch. After lunch she walked one mile before resting. At what point did she rest?" Answer: At K or X or H or Z. (A to D to K; A to D to X; A to R to H; A to R to Z.) As before, there isn't enough information given. You need to know the *direction* of each trip.

We might use arrows to indicate the 4 possibilities of the last illustration. Alice started at A –

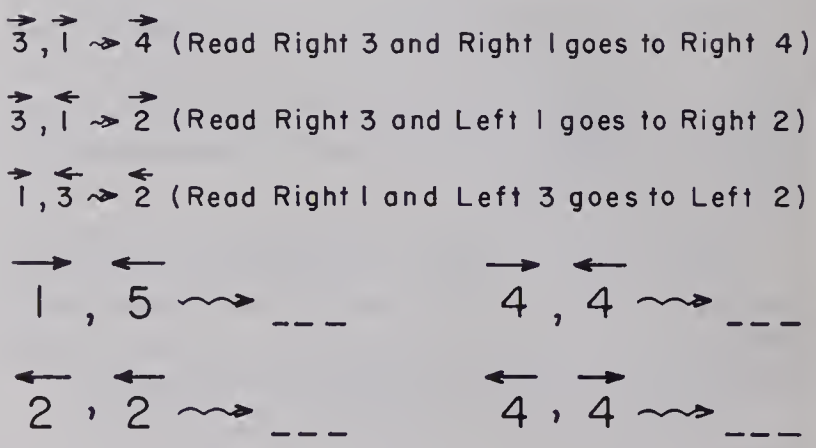
→ →
2, 1 takes her to H
→ ←
2, 1 takes her to Z
← ←
2, 1 takes her to K
← →
2, 1 takes her to X

All we need is an agreement about the meaning of the arrows. We might add some labels to our number line. One set of labels will be the same as before; the other will indicate the kind of single trip that would take her to that point from A:



"If she took no trip at all, she would go zero miles – a trip in no direction, so we will use 0 without any arrow for A in the second set of labels.

"Here are shorthand notes for trips Alice might take. Where would they take her if she started at 0?" (Read the wavering arrow as "goes to.")


3, 1 ⇝ 4 (Read Right 3 and Right 1 goes to Right 4)
3, 1 ⇝ 2 (Read Right 3 and Left 1 goes to Right 2)
1, 3 ⇝ 2 (Read Right 1 and Left 3 goes to Left 2)
1 , 5 ⇝ ___
4 , 4 ⇝ ___
2 , 2 ⇝ ___
4 , 4 ⇝ ___

Keep on making up examples until the two-way movement of the trips can be quickly visualized with the help of the number line.

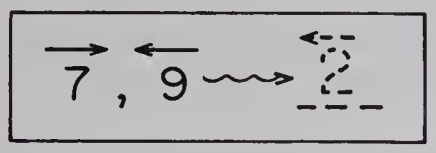
Turn now to page 68. Do not mention any parallelism between "The Countdown" and the activity outlined above. The parallelism will be evident to most everyone—and verbalization at this point will confuse rather than help. If arguments or dis-

agreements develop in filling in the blanks of the chart, ask each protagonist to draw a picture that supports his argument. The more highly visual the explanations are, the more convinced children will be later that "directed numbers" were invented because events in the real world indicated that some such device was necessary so arithmetic could more adequately express patterns that we notice around us.

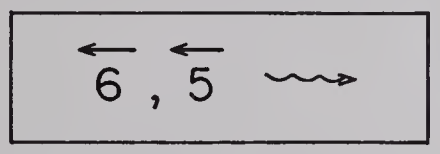
Notes to accompany *Work Text* page 69

Now, the emphasis is on a search through experience for ideas that directed numbers can help express . . . right, left; up, down; etc.

Use the "headline story" technique to explore these. Each story must explain the meaning of right arrow and left arrow.



"Right arrow means up the steps, and left means down the steps. I went up 7 stairs; then down 9, and ended up 2 steps below my starting point." Indicate the idea by completing the headline with a left arrow over 2.

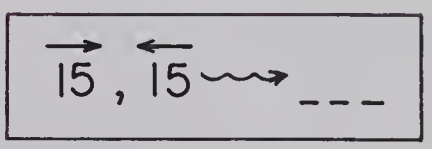


"Right arrow means a trip to the east, and left arrow means a trip in the opposite direction. We took a hike 6 miles west and rested. Then we went 5 miles further in the same direction. We ended up 11 miles west of our starting point. I can represent that idea in the headline by a left arrow over 11."

But some child may have another appropriate story. "Bill and Bob started at the same point. During the first hour and a half, Bill hiked 6 miles west and Bob hiked 5 miles. At that time they were 1 mile apart. Bill would have to hike 1 mile east to join Bob—or right arrow 1. On the other hand, Bob would have to hike one mile west to join Bill—or left arrow 1."

The meaning of the comma

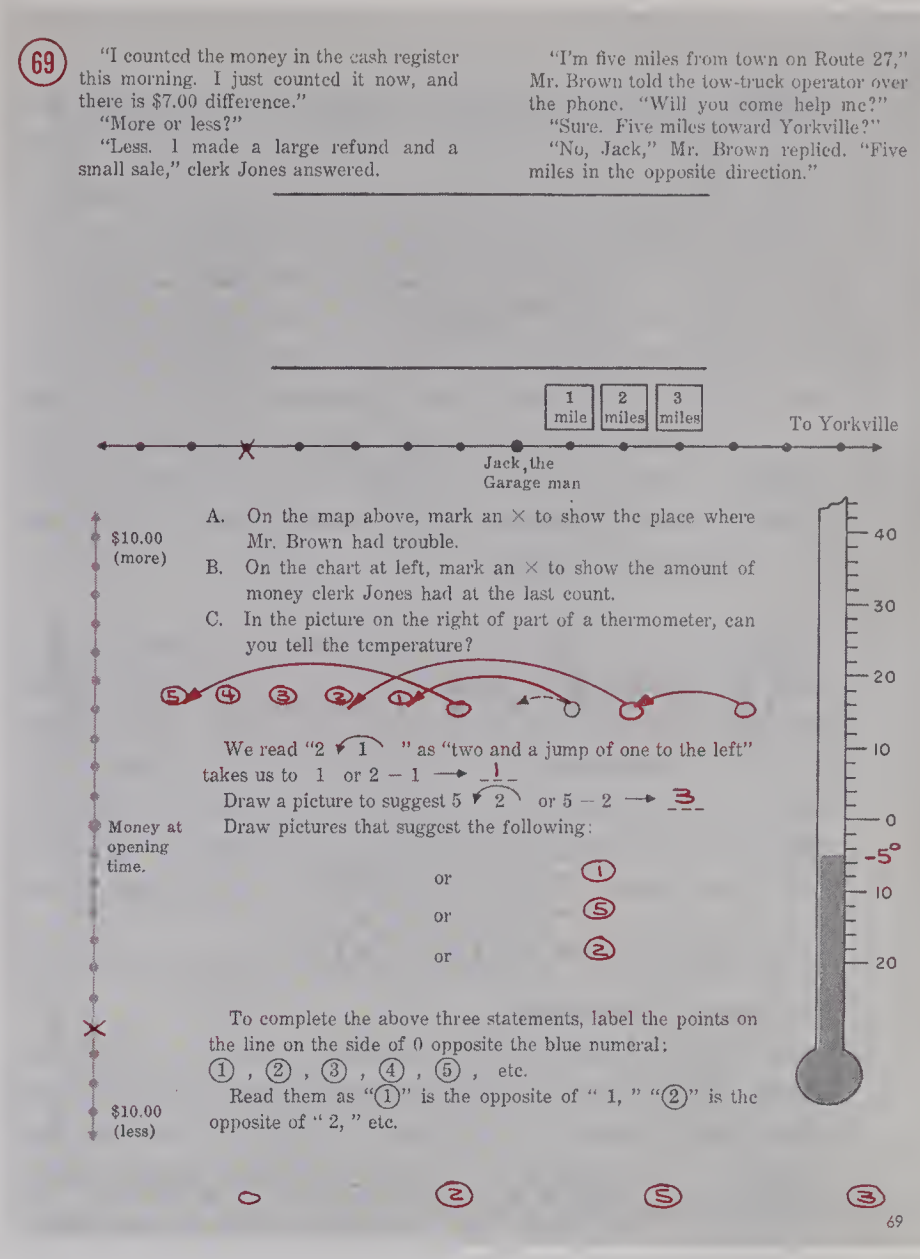
Why do we have three distinct interpretations? The answer is that we have no agreement about the meaning of the comma. We do not know what relationship between the two trips we want to consider. As we move along, this confusion ought to be eliminated—but let it be the subject of discussion, and do not discourage any reasonable argument.



"A right arrow means 'earned' and a left arrow means 'spent.' I earned $15 and then, on my vacation, I spent $15. So, I would complete the headline with 0."

Below the blue number line on page 69, an interpretation of "2 and a jump of 1 to the left" is suggested: 2 - 1 → 1. The 3 - 5 → ____ ends up 2 steps left of 0. The suggestion made here uses a different means if distinguishing the directions from 0—and the interpretation here leads to the conclusion that 3 - 5 → *2* (the opposite of "2").

Throughout the activity on page 69, and those outlined above, a great freedom of interpretation ought to be encouraged. There will be much use of "trips," "direction," "distance to right and left of 0," "above and below 0," etc. You may want to inject the idea of "opposite"—$\overrightarrow{5}$ and $\overleftarrow{5}$, or 5 and *five* are "opposites" with respect to 9, which is no longer the label that was always heretofore at one end of our labeling scheme. It has now become a dividing point in our schemes—a starting point for two very similar schemes that proceed in opposite directions.


69

"I counted the money in the cash register this morning. I just counted it now, and there is $7.00 difference."
"More or less?"
"Less. I made a large refund and a small sale," clerk Jones answered.

"I'm five miles from town on Route 27," Mr. Brown told the tow-truck operator over the phone. "Will you come help me?"
"Sure. Five miles toward Yorkville?"
"No, Jack," Mr. Brown replied. "Five miles in the opposite direction."

1 mile | 2 miles | 3 miles — To Yorkville
Jack, the Garage man

$10.00 (more)
Money at opening time.
$10.00 (less)

A. On the map above, mark an × to show the place where Mr. Brown had trouble.
B. On the chart at left, mark an × to show the amount of money clerk Jones had at the last count.
C. In the picture on the right of part of a thermometer, can you tell the temperature?

5 4 3 2 1

We read "2 ↶ 1" as "two and a jump of one to the left" takes us to 1 or 2 − 1 → 1
Draw a picture to suggest 5 ↶ 2 or 5 − 2 → 3
Draw pictures that suggest the following:

or ①
or ⑤
or ②

To complete the above three statements, label the points on the line on the side of 0 opposite the blue numeral:
① , ② , ③ , ④ , ⑤ , etc.
Read them as "①" is the opposite of " 1, " "②" is the opposite of " 2, " etc.

40 30 20 10 0 -5° 10 20

0 ② ⑤ ③

Notes to accompany *Work Text* page 70

Page 70 loses little time in reaching an agreement. Those numbers that behave like familiar whole numbers are written with no indication of direction —it is assumed. Numbers used to name points in the "opposite direction" from these familiar numbers will be referred to as "opposites" and indicated by a line added above the usual numeral. On a number line, these two sets of labels extend in opposite directions from 0. They may also be thought of as indicating trips.

While the black number line in the center of the page is the usual representation, three other number lines are shown, and any of the four can be used to help understand the examples on the page. You may want to mention that many people refer to these two systems of labels as "positive numbers" and "negative numbers" and "0" (which is not called either "negative" or "positive").

The sense of pattern

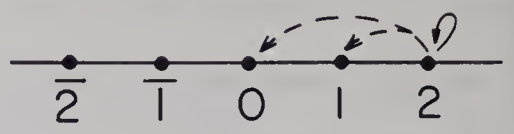
In the development of this and following pages, the *Work Text* depends almost exclusively upon the sense of pattern. A series of examples starts in the familiar, and by the time it reaches the unfamiliar, a pattern has begun that leads to a conclusion. Each time, consider this conclusion as tentative and test it for reasonableness. A most effective means of doing this is to employ the "headline story" technique. Also, sketch the idea behind conclusions on a number line.

Put the following examples on the board:

$2 - 0 \rightarrow 2$

$2 - 1 \rightarrow 1$

$2 - 2 \rightarrow 0$



Select one example as a headline:

$2 - 1 \rightarrow \_\_\_$

"It was 2° above zero and then fell 1°. The temperature was then 1° above zero."

"I was 2 miles north of our home. After walking south 1 mile, I was then 1 mile north of home."

Now, extend the series of examples into the unfamiliar:

A B

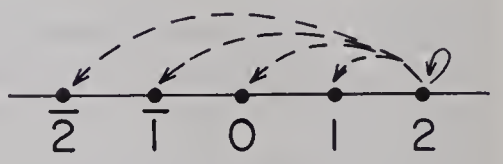
$2 - 0 \rightarrow 2$

$2 - 1 \rightarrow 1$

$2 - 2 \rightarrow 0$

$2 - 3 \rightarrow \overline{1}$

$2 - 4 \rightarrow \overline{2}$

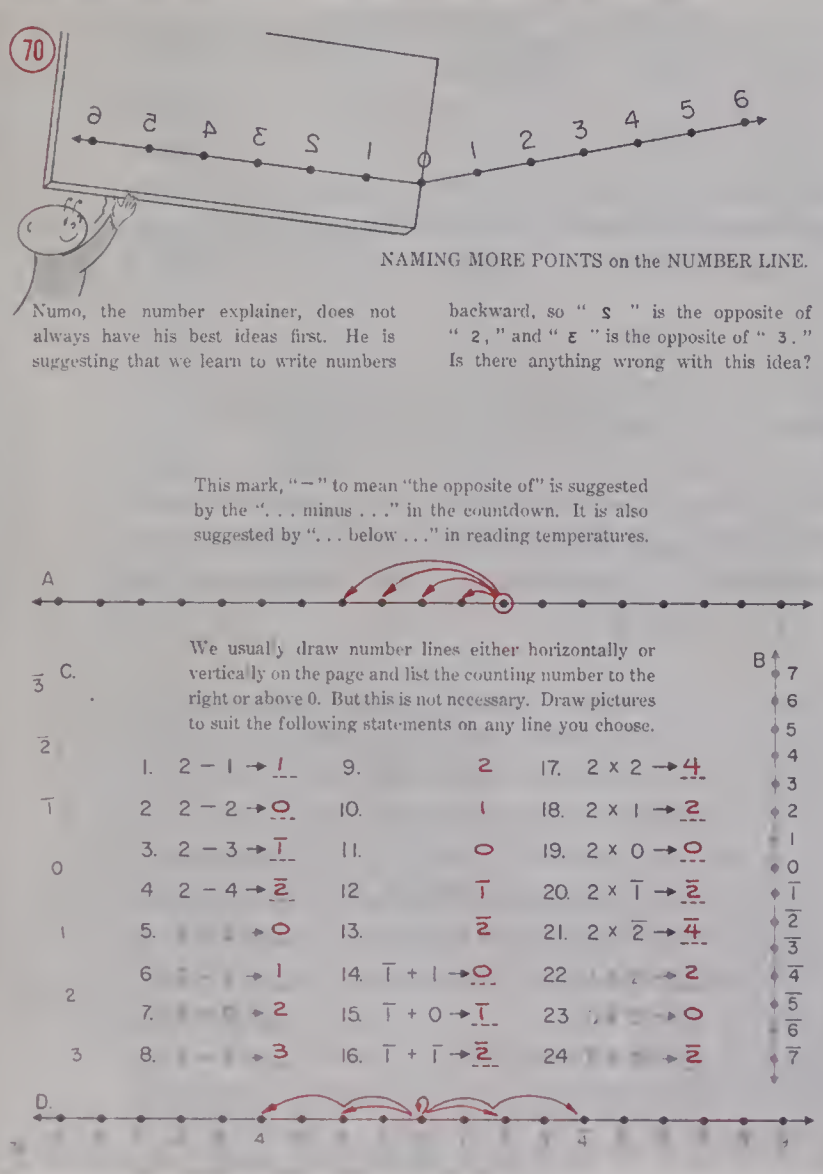


The series under arrow A is 0, 1, 2, 3, 4. The series under B starts out 2, 1, 0 and, by this time, we want to extend that series to 2, 1, 0, $\overline{1}$, $\overline{2}$. . . .At least a sense of pattern draws us in that direction. Try another headline:

$2 - 3 \rightarrow \overline{1}$

"It was 2° above zero and then fell 3°. The temperature was then 1° below zero."

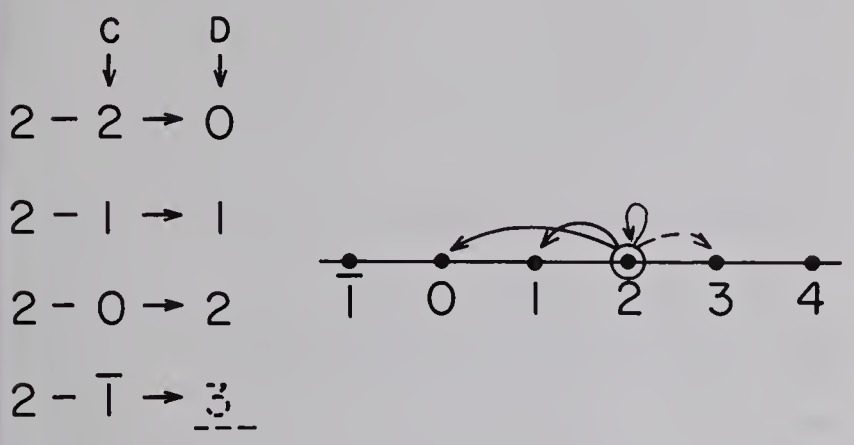
"I was 2 miles north of our home. After walking 3 miles south, I was then 1 mile south of home."


70
NAMING MORE POINTS on the NUMBER LINE.
Numo, the number explainer, does not always have his best ideas first. He is suggesting that we learn to write numbers backward, so " ƨ " is the opposite of " 2, " and " ε " is the opposite of " 3. " Is there anything wrong with this idea?
This mark, " − " to mean "the opposite of" is suggested by the ". . . minus . . ." in the countdown. It is also suggested by ". . . below . . ." in reading temperatures.
A
We usually draw number lines either horizontally or vertically on the page and list the counting number to the right or above 0. But this is not necessary. Draw pictures to suit the following statements on any line you choose.
B
C.
D.
1. 2 − 1 → 1
2 2 − 2 → 0
3. 2 − 3 → $\overline{1}$
4 2 − 4 → $\overline{2}$
5. → 0
6 → 1
7. → 2
8. → 3
9. 2
10. 1
11. 0
12 $\overline{1}$
13. $\overline{2}$
14. $\overline{1}$ + 1 → 0
15. $\overline{1}$ + 0 → $\overline{1}$
16. $\overline{1}$ + $\overline{1}$ → $\overline{2}$
17. 2 x 2 → 4
18. 2 x 1 → 2
19. 2 x 0 → 0
20. 2 x $\overline{1}$ → $\overline{2}$
21. 2 x $\overline{2}$ → $\overline{4}$
22 → 2
23 → 0
24 → $\overline{2}$

By retelling the stories that fit the other headlines, but changing them slightly to reflect the elements of the new headline, we are led to sensible conclusions.

The difficulty in example 8

Examples 5, 6 and 7 are easy to interpret, but example 8 offers some difficulty. Do not let the discussion become too argumentative until we encounter similar patterns in F, G, T and J on *Work Text* page 71. However, the difficulty ought to be faced:



The pattern in the sequence under arrow C is familiar enough. So is the sequence begun by the first three entries under D—0, 1 and 2. We are persuaded to extend that sequence to 0, 1, 2 and 3; but this leads us to the unfamiliar result that $2 - \bar{1} = 3$.

Try a couple of headlines:

2 − 1 → 1

"Bill had $2. He spent $1, and had $1 left."

"From 2° above zero, the temperature fell 1°. It was then 1° above zero."

Now to the more puzzling headline:

2 − $\bar{1}$ → 3

"Bill had $2. Mary asked him how much he spent. 'The opposite of $1,' was Bill's answer. He must mean that, instead of spending $1, he earned $1 . . . so he now had $3."

"'How far has the temperature fallen in the last hour?' Mother asked. 'The opposite of 1°' was Mary's answer. The temperature must have gone up 1°, to a reading of 3° above zero."

Another puzzler is to find sensible interpretations for the last two in the group of examples 17, 18, 19, 20, 21. The pattern leads us to write:

20. $2 \times \bar{1} \rightarrow \bar{2}$

21. $2 \times \bar{2} \rightarrow \bar{4}$

$2 \times \bar{2} \rightarrow \bar{4}$

"For 2 hours the temperature fell 2° each hour. During that 2-hour period, the temperature fell 4°."

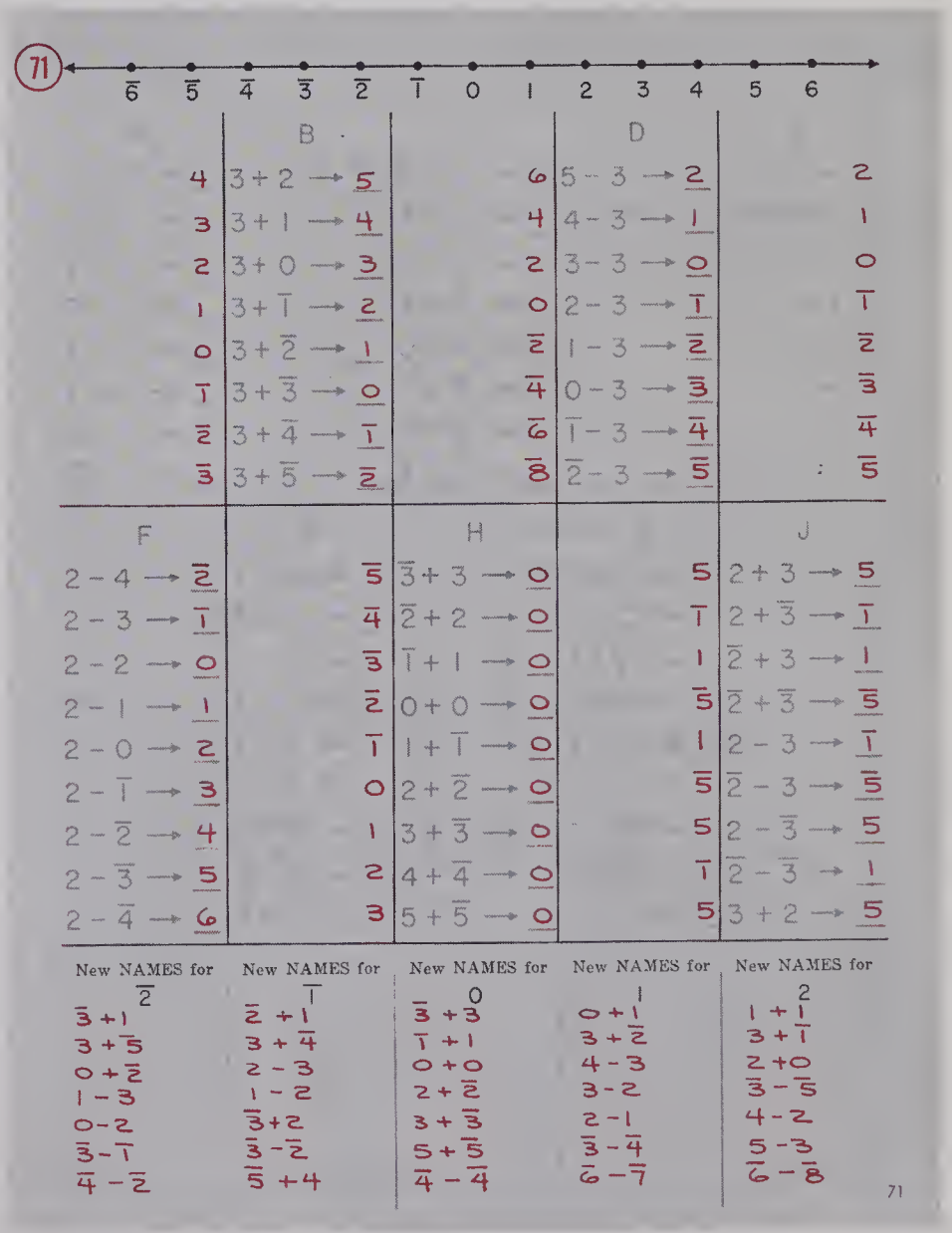
"Twice last week I lost a pair of pencils."

Again, do not labor these examples, but be ready to suggest such interpretations.

Notes to accompany *Work Text* page 71

"Let pattern be your guide on page 71 in columns A through h. When in doubt, extend any pattern that has come to view. Columns T and J require a different approach."

With such advice, the blanks ought to be filled



in quickly. If there is hesitation, direct attention to the labels on the numberline at the top of the page. Before considering columns T and J, suggest that some of the expressions be used as headlines.

$$\overline{2} + \overline{4} \rightarrow \overline{6}$$

"On the first page of a test, I made 2 mistakes. On the second page, I made twice as many. That's a total of 6 mistakes."

$$\overline{2} - 3 \rightarrow \overline{5}$$

"I started out owing a 2¢ fine at the library. Then I lost 3¢. Now I'm 5¢ worse off."

$$3 + \overline{5} \rightarrow \overline{2}$$

"I needed to jump 3 feet to get over the wall. When I jumped down on the other side, it was a 5 foot jump. The ground was 2 feet lower on that side than the side I started from."

$$3 + \overline{3} \rightarrow 0$$

"We drove 3 miles, then we drove 3 miles in the opposite direction and were right back where we started—just as if we had gone nowhere at all."

$$\overline{3} - \overline{3} \rightarrow 0$$

"Any number subtracted from itself is zero. That's like having a debt of $3 and someone takes it away. You are better off by three dollars, but you have nothing more than a load off your shoulders."

"I dug a hole 3 feet deep. Then I did the opposite of digging a hole 3 feet deep—I filled it up."

This example helps give a check point in the unfamiliar part of the sequence under G:

$\overline{3} - 2 \rightarrow \overline{5}$
$\overline{3} - 1 \rightarrow \overline{4}$
$\overline{3} - 0 \rightarrow \overline{3}$
$\overline{3} - \overline{1} \rightarrow$ ____
$3 - \overline{2} \rightarrow$ ____
$3 - \overline{3} \rightarrow$ ____

Any number subtraced from itself is zero. This is a strong argument that extending the pattern of $\overline{5}$, $\overline{4}$ and $\overline{3}$ will lead us to the reasonable statement further on that $3 - \overline{3} = 0$.

Discovery leads to understanding

In columns T and J, a start can be made by looking back to identical examples in earlier columns. In these two columns, all but four in each are repetitions, and some of the rest will cause no difficulty. The last example in column T can be written along with three somewhat similar expressions:

$3 + 2 \rightarrow 5$
$3 + \overline{2} \rightarrow 1$
$3 - 2 \rightarrow 1$
$3 - \overline{2} \rightarrow$ ____

We are tempted to write: $3 - \overline{2} \rightarrow 5$.

Other puzzles can be treated similarly. Throughout this discussion keep insisting that we are confronted with the problem of trying to find ways in which these new numbers can be made to behave sensibly. Keep asking whether they help point to patterns we find in real life. There is no question of "right" and "wrong." And, most of all, beware of "rules" that do not force themselves on children as sensible interpretations are tested. If "rules" are suggested by individuals, direct the class to carry out exhaustive tests of their assertions.

Remember that to advance "rules" at this time will forever bar the children from discovering sensible ways for interpreting these numbers—and discovery is the only sure road to understanding.

Notes to accompany *Work Text* page 72

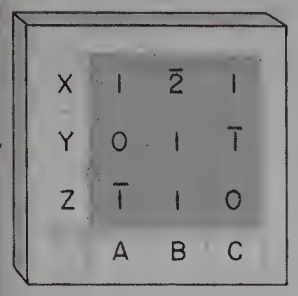
Put a replica of the playing board on the blackboard, along with a number line as shown on page 72. Then select two teams—an ABC team and an XYZ team. Arbitrarily, team XYZ is assigned the task of reaching $\overline{5}$ and ABC is assigned the task of reaching 5 (the opposite of the opposite of 5).

Whispering in a huddle, each team selects one of its letters and the captain writes it down.

The captain now selects the row chosen (by XYZ) and the column chosen (by ABC). Suppose the selections are Y and C. A number is noted in row Y and column C—it is $\overline{1}$.

Now a scorekeeper is needed to keep a record on the number line. This first play takes the score to $\overline{1}$. Now XYZ has only 4 more to go, but ABC has 6 to go.

Here is the playing board:



Jim

Jim and Mary are playing.

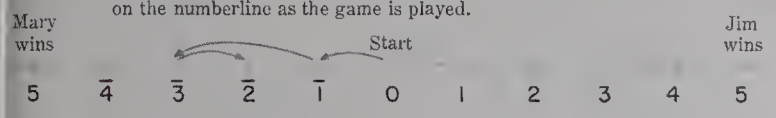
Each turn, Jim picks a letter on his side of the board (A, B, or C.) He writes it down. Mary cannot look at Jim's choice.

Then Mary picks a letter on her side of the board (X, Y, or Z).

The score for the turn is found where the row picked by Jim and the column picked by Mary cross.

Example: Jim chooses A, and Mary chooses Z. The score is $\bar{1}$ — "the opposite of one."

Put a marker on the starting point. Mary wins if the counter reaches $\bar{5}$. Jim wins if the counter reaches 5. Move the counter on the numberline as the game is played.



The score, after the example, is $\bar{1}$. Who is ahead?

On the next turn, Jim chooses B, and Mary chooses X. (B, X → $\bar{2}$)

Now the total score is $\bar{3}$. Who is ahead?

On the next play, Jim chooses A, and Mary chooses X again. (A, X → 1)

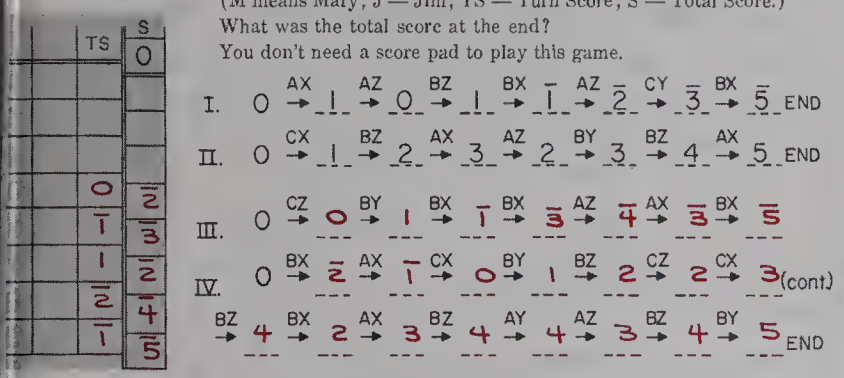
Now the total score is $\bar{2}$.

That helped Jim, but Mary is still ahead.

(M means Mary; J — Jim; TS — Turn Score; S — Total Score.)

What was the total score at the end?

You don't need a score pad to play this game.


TS
S
0
0 1̄
1 2̄
2̄ 4
4 5
I. 0 AX→ 1 AZ→ 0 BZ→ 1 BX→ 1̄ AZ→ 2̄ CY→ 3̄ BX→ 5̄ END
II. 0 CX→ 1 BZ→ 2 AX→ 3 AZ→ 2 BY→ 3 BZ→ 4 AX→ 5 END
III. 0 CZ→ 0 BY→ 1 BX→ 1̄ BX→ 3̄ AZ→ 4̄ AX→ 3̄ BX→ 5̄
IV. 0 BX→ 2̄ AX→ 1̄ CX→ 0 BY→ 1 BZ→ 2 CZ→ 2 CX→ 3 (cont.)
BZ→ 4 BX→ 2 AX→ 3 BZ→ 4 AY→ 4 AZ→ 3 BZ→ 4 BY→ 5 END

Another huddle in whispers, and each team chooses once again one of the letters in its name. Suppose they select Z and C. What number is noted in row Z and column C? The score indicated is 0.

So, no change is noted on the scoreboard. The total is still $\bar{1}$ (the opposite of 1), and XYZ still has the advantage.

Thus the game proceeds, round by round, until the score reaches 5 or $\bar{5}$, and one team or the other wins.

After the first game, ask the question: "Which team has the best opportunity to win—XYZ or ABC?" (The fact of the matter is that each has the same chance of winning. The sum of each row and each column is 0: $1 + \bar{2} + 1 = 0$; $0 + 1 + \bar{1} = 0$; $\bar{1} + 1 + 0 = 0$; $\bar{1} + 0 + 1 = 0$; $\bar{2} + 1 + 1 = 0$; $1 + \bar{1} + 0 = 0$.)

After several games, *Work Text* page 72 and the top of *Work Text* page 73 will offer little difficulty.

Notes to accompany *Work Text* page 73

The "headline story" activity on page 73 brings us back for another look at interpretations for this expanded number system.

Have the children complete all four headlines in example 1. Discuss the merits of each as a title for the story given. When they agree on one, put it on the board.

$\bar{3} + 5 \rightarrow 2$

Then ask the class to find other situations that would fit this headline.

"I took 3 steps backward and 5 steps forward. Now I am 2 steps ahead of where I was."

"The temperature rose 5° after starting at 3° below zero. Now the temperature is 2° above zero."

"I had a debt of $3. After earning $5, I paid off my debt and have $2 left."

"Mary broke 3 eggs by accident. She threw them out and put 5 unbroken eggs in the box. Now there are 2 more unbroken eggs than before."

"A river was flowing at the rate of 3 miles per hour. John was rowing a boat upstream at the rate of 5 miles per hour. It took him an hour to go 2 miles upstream—or half an hour to go 1 mile."

Scotty Brown thought this game was too slow, so he invented

The Game of Double Opposites.

That's the only change. Here are Sample games.

1. 0 CY→ 2̄ AZ→ 4̄ BY→ 2̄ BX→ 6̄ - END
2. 0 BZ→ 2 BY→ 4 CZ→ 4 CX→ 6 - END

Ellen Burke wanted to speed the game up in another way so she invented

The Short Game of Opposites

Ellen's Rule:

Select the plays so the games end as shown.

3. 0 → ___ → ___ → ___ → ___ - END
4. 0 → ___ → ___ - END

Complete the headlines and check one that best fits each story.

1. Bill thought: "We are at 3 minutes before blast-off time. Where will we be in 5 minutes?"

$\bar{3} + \bar{5} \rightarrow \bar{8}$ | $\bar{3} + 5 \rightarrow 2$ ✓ | $\bar{3} + 8 \rightarrow 5$ | $\bar{2} + \bar{3} \rightarrow \bar{5}$

2. I lost a nickel and found a dime.

$\bar{5} + 15 \rightarrow 10$ | $5 - 10 \rightarrow \bar{5}$ | $\bar{5} + \bar{10} \rightarrow \bar{15}$ | $\bar{5} + 10 \rightarrow 5$ ✓

3. After I opened the store, Mrs. Jones returned a $4.00 book and bought a $3.00 book instead.

$\bar{3} + 2 \rightarrow \bar{1}$ | $\bar{4} + 3 \rightarrow \bar{1}$ ✓ | $\bar{4} + \bar{3} \rightarrow \bar{7}$ | $10 + \bar{3} \rightarrow 7$

4. The temperature was 5 degrees below zero and fell to 8 degrees below zero.

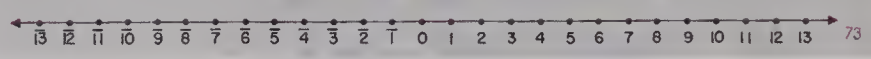
$\bar{5} + \bar{8} \rightarrow \bar{13}$ | $\bar{5} - \bar{3} \rightarrow \bar{8}$ ✓ | $\bar{5} - 8 \rightarrow \bar{13}$ | $5 - 8 \rightarrow \bar{3}$

5. Meg and Polly each lived 4 blocks from school, but in opposite directions.

$\bar{4} + \bar{4} \rightarrow \bar{8}$ | $2 - 4 \rightarrow \bar{2}$ | $\bar{4} + 8 \rightarrow 4$ ✓ | $8 + \bar{8} \rightarrow 0$

6. Jerry dug a hole 10 feet deep. He put a 12 foot ladder in the hole. How much of the ladder was above ground.

$\bar{10} + 12 \rightarrow 2$ ✓ | $10 + 12 \rightarrow 22$ | $\bar{10} + \bar{12} \rightarrow \bar{22}$ | $\bar{12} + 14 \rightarrow 2$



"Our baseball team was 3 runs behind at the beginning of the fourth inning. The other team didn't get any runs, but we made 5 runs. At that point, we were 2 runs ahead."

With each of the six examples on page 73, allow as much discussion as there is time for about the selection of the title itself, and about other stories for the same headline. Our main concern at this point is not at all with manipulation of these new numbers. Instead, it is with understanding that these numbers were invented because there are so many situations in real life with the same pattern—situations in which direction plays a decisive role in understanding the outcome.

The headline for example 5

Selection of the most appropriate headline in example 5 may lead to a feeling that none is as appropriate as $\overline{4} + 4 \rightarrow 0$, or some other expression. However, among those given, the most qualified candidate is:

$$\overline{4} + 8 \rightarrow 4$$

"Meg lived 4 blocks west of school. She must walk 8 blocks east to visit Polly, because Polly lives 4 blocks east of school."

Other appropriate stories that fit this headline?

"I spent $4 and earned $8. I now have $4 more than I had before."

"Mr. Jones went to work at 4 hours before midnight and he worked for 8 hours. He finished at 4 A. M., or 4 hours after midnight."

"There were 8 minutes between 4 minutes before blastoff and 4 minutes after blastoff."

"From home, I walked 4 blocks south to the store, then 8 blocks north to the library. The library is 4 blocks north of my house."

Remember, the main concern at this time is understanding how these new numbers help express patterns of events in the world around us.

Notes to accompany *Work Text* page 74

We have been using an arrow to indicate the outcome when two numbers are considered in certain ways. Let us, at this time, consider the question: "Do you think we can replace the 'goes to' arrow with an equal sign?"

Which suggests this prompter: "When have we been using the expression 'equal' in arithmetic?" (Answer: to indicate that we have two names for the same number.)

Now, refer to the bottom of *Work Text* page 71. In each column, new names for each of five numbers were written. "Can we write a sentence by rewriting any pair from any column and writing an equal sign between them?" Consider them column by column:

$$2 - 4 = \overline{2}$$

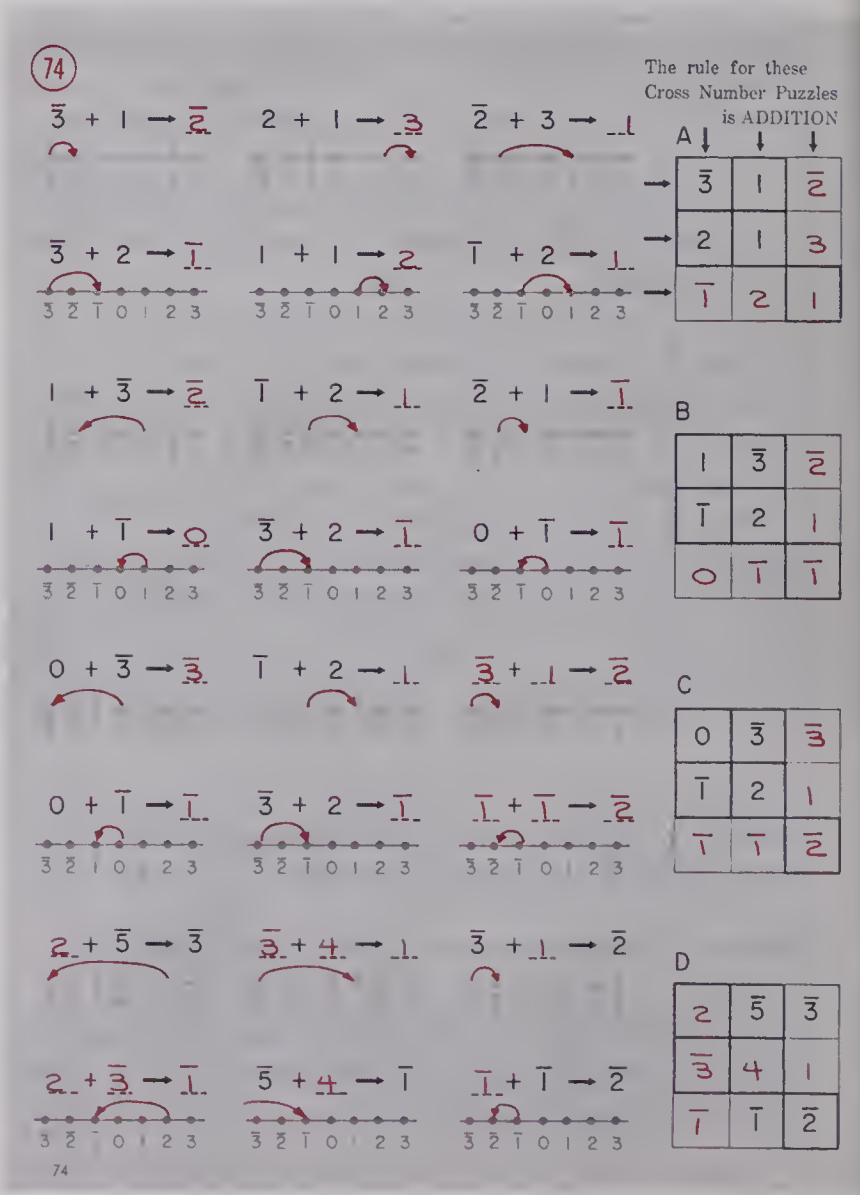
$$1 - 3 = \overline{2}$$

$$\overline{2} = \overline{5} + 3$$

"As we work through pages 74 and 75, we will keep writing the arrow. But, in each example, ask yourself whether or not the arrow stands between two names for the same number."

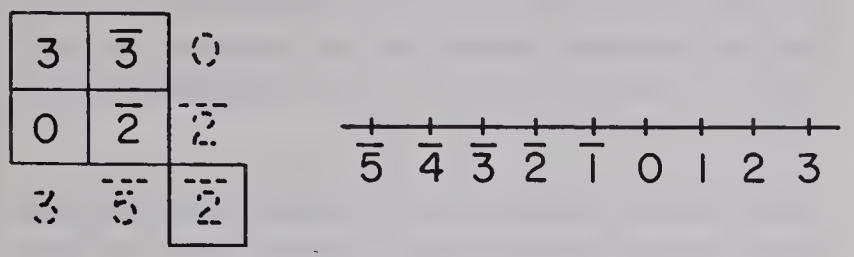
On page 74 there are four Cross Number Puzzles. The rule to be followed in each is noted at the top—"addition."

"Do you think the Cross Number Puzzle with the rule of addition will 'work' if we use numbers



and the opposites of numbers? Let's try some examples."

To the left of each Cross Number Puzzle, six chunks of number lines are provided so that a sketch can be drawn to suggest the idea of each step leading toward the check box in the lower right-hand corner. Try an example on the board:



Use the number line to reach a conclusion in each step. Then refer to page 71. Look for a similar example. Are the conclusions the same?

$3 + \overline{3} \rightarrow 0$ (see column H on page 71)
$0 + \overline{2} \rightarrow \overline{2}$ (see column C on page 71)
$0 + \overline{2} \rightarrow \overline{2}$ (see column C on page 71)

$3 + 0 \rightarrow 3$ (see column B on page 71)
$3 + \overline{2} \rightarrow \overline{5}$ (see column T on page 71)
$\overline{3} + \overline{5} \rightarrow \overline{2}$ (see column B on page 71)

Again, our conclusions are reinforced as we arrive at the same results in another context . . . and we have not yet encountered an example of inconsistency.

"Does the arrow stand, in each case, between different names for the same number?"

The Cross Number Puzzle as headline

Using the Cross Number Puzzle above as a headline, can we think of an appropriate story?

"Consider the initial four cells as a place to keep score of the situation in each of four quarters of a basketball game. In one quarter, we scored 3 more points than our opponents. In another quarter, they scored 3 more than we did (3). In another, we scored the same number of points. In the other quarter, they outscored us by 2 points (2). Who won the game? By how many points was the game won?

"Suppose the top row represents the scoring situation in the first half—the first 2 quarters. We moved ahead 3 points in the first quarter and scored 3 less than our opponents in the second quarter. The score would be tied—neither side in the lead—the scores are separated by zero points.

"In the first quarter of the second half, both sides scored the same number of points, but in the last quarter, the opponents outscored us by 2 points—or, in the entire second half, they gained on us by 2 points. Finally, since we were tied at the end of the first half, and they outscored us by 2 points in the last half, they won the game by 2 points."

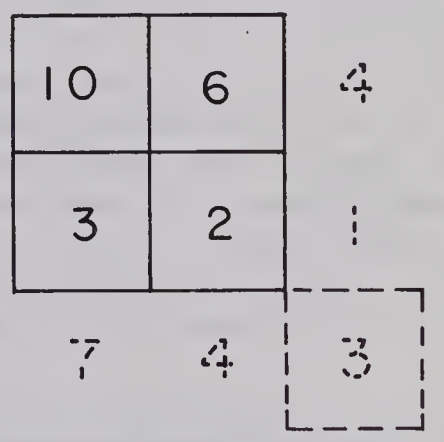
Suppose at this point, someone breaks in with, "I was the scorekeeper, and you don't remember the game too well. I kept the results of the first half in the left hand column, and we were ahead at the half by 3 points. In the second half—recorded in the column on the right—they outscored us by 3 points in one quarter and by 2 points in the final quarter, or by a total of 5 points in the last half. Since we were ahead by only 3 points at the half and they got 5 more than we did in the second half, they won by 2 points."

So it seems that the In-Any-Order Rule for addition applies to our expanded system of numbers as well as it did before.

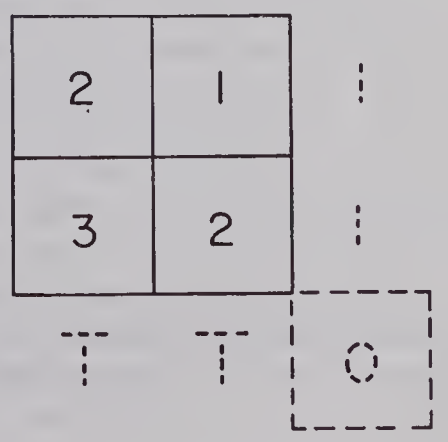
If any difficulty arises in the four Cross Number Puzzles on *Work Text* page 74, consider each as a scoreboard of a basketball game as we did the example above.

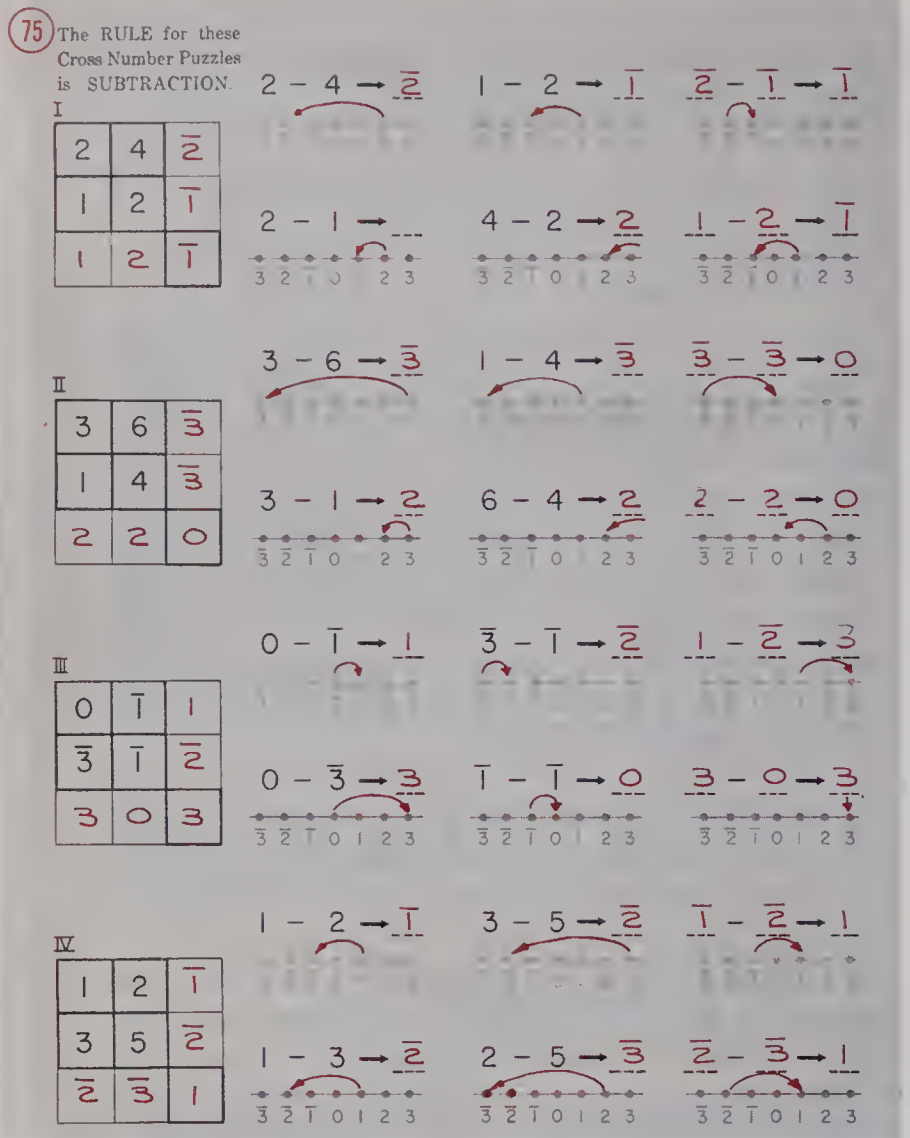
Notes to accompany *Work Text* page 75

Anticipate page 75 with an example of a Cross Number Puzzle that does not introduce "opposites" of numbers and illustrates the rules of "subtraction" as they apply in this form of puzzle:



"Suppose we try one not so carefully planned."




75 The RULE for these Cross Number Puzzles is SUBTRACTION.
I
II
III
IV
2 − 4 → $\overline{2}$
1 − 2 → $\overline{1}$
$\overline{2}$ − $\overline{1}$ → $\overline{1}$
2 − 1 →
4 − 2 → $\overline{2}$
1 − $\overline{2}$ → $\overline{1}$
3 − 6 → $\overline{3}$
1 − 4 → $\overline{3}$
$\overline{3}$ − $\overline{3}$ → 0
3 − 1 → 2
6 − 4 → $\overline{2}$
$\overline{2}$ − $\overline{2}$ → 0
0 − $\overline{1}$ → 1
$\overline{3}$ − $\overline{1}$ → $\overline{2}$
1 − $\overline{2}$ → 3
0 − $\overline{3}$ → 3
$\overline{1}$ − $\overline{1}$ → 0
3 − 0 → 3
1 − 2 → $\overline{1}$
3 − 5 → $\overline{2}$
$\overline{1}$ − $\overline{2}$ → 1
1 − 3 → $\overline{2}$
2 − 5 → $\overline{3}$
$\overline{2}$ − $\overline{3}$ → 1
$\overline{3}$ $\overline{2}$ $\overline{1}$ 0 1 2 3
75

Each of the steps in this example can be found on page 71, except the fact that $\overline{1} - \overline{1} = 0$.

$2 - 1 \rightarrow 1$ $\qquad$ $2 - 3 \rightarrow \overline{1}$
$3 - 2 \rightarrow 1$ $\qquad$ $1 - 2 \rightarrow \overline{1}$
$1 - 1 \rightarrow 0$ $\qquad$ $\overline{1} - \overline{1} \rightarrow 0$

"Does the arrow, in each case, stand between two names for the same number? Whenever we subtract any number from itself, what number have we named? (Answer: 0.) Can we argue that $\overline{1} - \overline{1} = 0$? (Answer: Yes.)

"Can we use this Cross Number Puzzle as a headline?

"In the top row, let's record the fact that Bill started with 2 dollars and spent $1, so he had $1 left. In the second row, we record the fact that Jim started with $3 and spent $2. He had $1 left. Who had more money left? Neither, because $1 - 1 = 0$."

But someone has this to say: "You interpreted the record in the wrong way. Bill earned $2 and spent $3, as the left hand column indicates, so Bill was $1 worse off than before. Jim earned only $1 and spent only $2. He was then $1 worse off than before.

"Now the question is—was Jim better off than Bill at the end, or was Bill better off? The answer is that they were the same amount behind—there was no difference between their sad conditions—the difference between their debts was $0."

The reason for the variance

There certainly is a wide difference in facts in these two stories. In one Bill started out with more than Jim; in the other, he started out with less. In one, both boys came through with $1 a piece; in the other, they were each worse off by $1.

Such a wide difference in interpretation suggests that there may be a significant difference attached to the order in which we subtract.

$(7-4) - (2-1) = 3 - 1 = 2$
$(7-1) - (4-2) = 6 - 2 = 4$

However, the Cross Number Puzzle works because there is a carefully controlled change in order. For example, in the first example above, the following order change occurred:

$(10-6) - (3-2) = 4 - 1 = 3$
$(10-3) - (6-2) = 7 - 4 = 3$

Each step taken in each puzzle on page 75 is checked on a number line to the right of the example.

After completing the page, return to the discussion of the arrow: "Does the arrow, in each case, stand between two names for the same number? (Answer: Yes.) Then you will not be surprised that an 'equal' sign replaces the arrows on the *Work Text* pages that follow."

Notes to accompany *Work Text* page 76

Page 76 approaches the same material covered in preceding pages, but from two new points of view. The first is based squarely on the idea of expanding patterns. The second considers six fact teams, rearranging the sum and the two addends.

The results of both approaches are consistent with all previous work involving numbers and their opposites. We have not mentioned rules, but have kept an eye on testing results in terms of reasonableness.

Nothing can be gained at this point by talking about rules. We will direct our attention to other matters after *Work Text* pages 76 and 77. All that the children will remember for long is what they *understand.*

Appeal to pattern

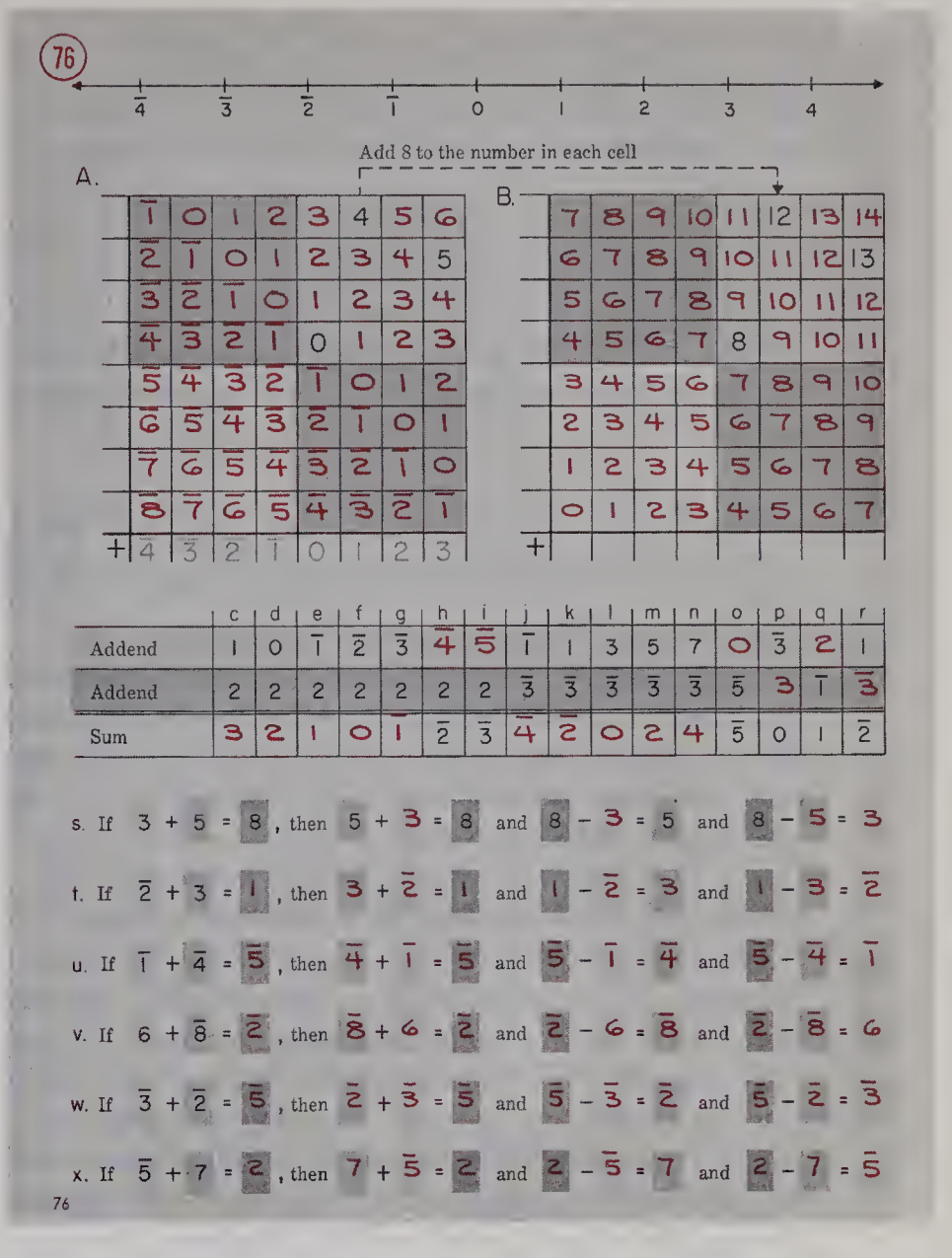
In example A, fill in the area tinted blue so that each cell contains the sum of the addends noted at the left side of each row and the bottom of each column. The result is a 4 by 4 corner of an adding machine.

Before going on with A, move to B and fill in the cells in the blue-tinted area—each with a number that is 8 more than the number noted in the corresponding cell of A.

Notice the sequences started in the blue area of B. The right-hand column, beginning at the top, reads 14, 13, 12, 11. If you were to expand on that pattern, you would keep on going down the column, writing 10, 9, 8, 7, 6. Complete the column in that way.

Complete all four columns by extending the pattern that is evident in that part of each which is in the blue-tinted area.

Next, consider the rows in B that, from right to left, commence in the blue-tinted area. Extend the sequences. Now, use the same approach to complete the four columns on the left that begin at the top in the green-tinted area.


76
Add 8 to the number in each cell
A.
B.
Addend
Addend
Sum
s. If 3 + 5 = 8, then 5 + 3 = 8 and 8 − 3 = 5 and 8 − 5 = 3
t. If 2̄ + 3 = 1, then 3 + 2̄ = 1 and 1 − 2̄ = 3 and 1 − 3 = 2̄
u. If 1̄ + 4̄ = 5̄, then 4̄ + 1̄ = 5̄ and 5̄ − 1̄ = 4̄ and 5̄ − 4̄ = 1̄
v. If 6 + 8̄ = 2̄, then 8̄ + 6 = 2̄ and 2̄ − 6 = 8̄ and 2̄ − 8̄ = 6
w. If 3̄ + 2̄ = 5̄, then 2̄ + 3̄ = 5̄ and 5̄ − 3̄ = 2̄ and 5̄ − 2̄ = 3̄
x. If 5̄ + 7 = 2, then 7 + 5̄ = 2 and 2 − 5̄ = 7 and 2 − 7 = 5̄

"What about the bottom 4 rows? Do they fit into the patterns we have been following?" They do.

Move back to example A. The column on the right, from the top down, reads 6, 5, 4 and 3. It should be extended to include 2, 1 and 0. That leaves one cell in the bottom row not labeled.

"Look at the number line above. Do you see a sequence of labels that is part of the one we are considering in the right-hand column?" Yes—4, 3, 2, 1 and 0. "On the number line, what would extend the sequence?" Answer: the opposite of one, or $\overline{1}$.

Discuss the next three columns that include the blue section. Then consider the four rows that include the blue section. "Can we use the same method to extend the left-hand columns to the bottom?" And then, of course, we notice that the bottom rows fall nicely into the pattern.

Notice the diagonals, top down to the right. Each is a repetition of the same number.

Notice the diagonals, top down to the left. The main diagonal reads 6, 4, 2, 0, $\overline{2}$, $\overline{4}$, $\overline{6}$, and $\overline{8}$. The diagonal next to it reads 5, 3, 1, $\overline{1}$, $\overline{3}$, $\overline{5}$, and $\overline{7}$.

Select one of the blue numerals and one of the green numerals. Consider the number names as addends. Note the number written where the row selected crosses the column selected. Is that number the sum of the addends?

There can be no question when numbers noted in the blue area are under examination. "What addends indicate 0 as their sum?" (Any number added to its opposite indicates 0.)

After many examples are considered, turn to the relationship between A and B. We hope that the numbers noted in each cell of B is 8 more than the number noted in the corresponding cell of A.

In the bottom left-hand corner of A, we have written $\overline{8}$. In the corresponding cell of B, we have noted 0. Is it true that $\overline{8} + 8 = 0$? Try many similar situations based on the relationship between corresponding cells.

Across the middle of the page, a chart is given that can be used to record results from the development above.

Fact teams

Examples S through X study the idea of a fact team—as four ways to express the same idea. The first is most familiar:

If 3 + 5 = 8, then 5 + 3 = 8
and 8 − 3 = 5 and 8 − 5 = 3

"In each member of the fact team, point out the sum and point out the addends.

"If numbers and their opposites behave in a familiar way, then example *t* ought to be completed to read as follows:"

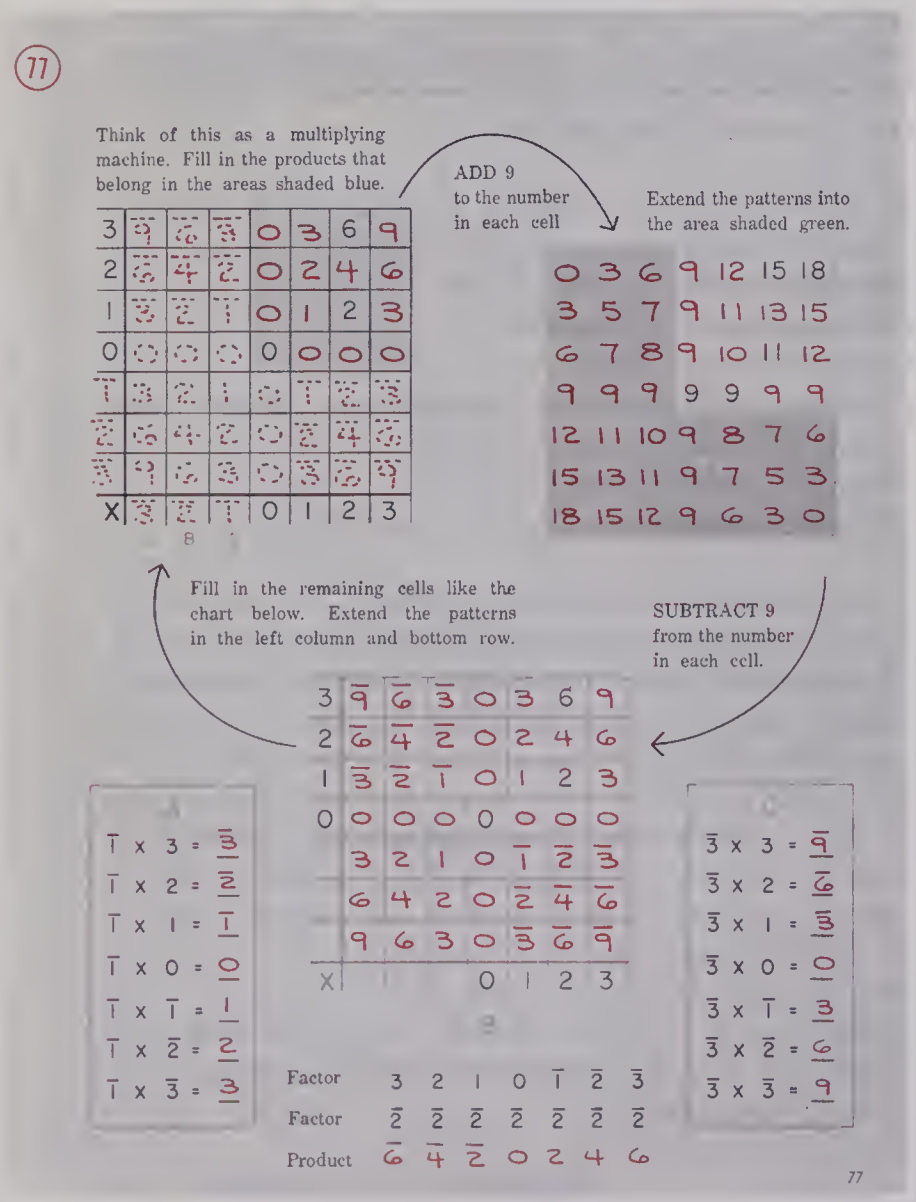
St D E E St D

If $\bar{2}+3=1$, then $3+\bar{2}=1$ and $1-3=\bar{2}$ and $1-\bar{2}=3$

As each example is developed, we come back again to the same results we have been considering since the Countdown on page 68. And in all this work, we have not encountered a single example that looks like a flaw in the pattern.

Notes to accompany *Work Text* page 77

Consider first the blue area of the "Multiplying Machine" in the upper left-hand corner of *Work Text* page 77. The numbers noted at the bottom and the left are factors. Select a factor from each group. Write the product in the cell common to both the row and the column selected. This is the way "Multiplying Machines" have been built before.


77
Think of this as a multiplying machine. Fill in the products that belong in the areas shaded blue.
ADD 9 to the number in each cell
Extend the patterns into the area shaded green.
Fill in the remaining cells like the chart below. Extend the patterns in the left column and bottom row.
SUBTRACT 9 from the number in each cell.
Factor
Factor
Product
77

When the 16 cells in the blue area are completed, add 9 to each of the products already found and note the result in the corresponding cells of the chart on the right—in the 16 unshaded cells.

In each column and row, there is a pattern begun that could be extended. The columns from top to bottom are:

18, 15, 12, 9, ____, ____, ____, ____
15, 13, 11, 9, ____, ____, ____, ____
12, 11, 10, 9, ____, ____, ____, ____
9, 9, 9, 9, ____, ____, ____, ____

Extend each pattern and record the results in the green-tinted cells of each column. Each row, from right to left, begins with the same sequence. Complete the top four rows in the same way as the columns were completed. Now, consider the array of 4 rows of 3 green cells filled in the top left hand corner:

0	3	6	
3	5	7	
6	7	8	
9	9	9	

Reading down each column, a pattern is noticeable that can be extended in each:

0, 3, 6, 9, ____, ____, ____
3, 5, 7, 9, ____, ____, ____
6, 7, 8, 9, ____, ____, ____

The extensions complete the bottom three rows, and in each there is an unbroken pattern.

"Remember, we noted in the upper right hand corner—the unshaded area—the numbers that were 9 more than the numbers noted in corresponding cells of the first grid.

"In the unshaded grid of green lines, indicate $\bar{9}$ less than the number noted in corresponding cells of the grid above and to the right.

"This chart is identical with the original one in the top left corner, insofar as the original is filled in. Complete the original so that both are alike.

"Notice that the bottom list of factors (in black) and the list of factors on the left (in black) are a sequence: 3, 2, 1, and 0. Continue this pattern—3, 2, 1, 0, $\overline{1}$, $\overline{2}$, $\overline{3}$—as the number-line sequence reads.

"Does this lead us to any unusual results? The column above A would indicate that $\overline{1} \times 3 = \overline{3}$, etc. Notice the green box labeled A and complete the sentence based on the Multiplying Machine.

"Do the same with the columns above B and C, recording the results in charts B and C."

Now to headlines

"Can you think of real situations that might fit headlines made of some of these statements?"

$$3 \times \overline{2} = \overline{6}$$

"The temperature dropped 2° each hour for 3 hours. In that time the temperature dropped 6°."

"I went down 2 steps at a time. In 3 strides I was 6 steps below my starting place."

$$\overline{2} \times 3 = \overline{6}$$

Of course, the stories that fit the headline in the example above fit this one equally as well.

"On the first day, I dug a hole 2 feet deep; on the next day, I dug it 2 feet deeper; and 2 feet deeper on the third day. By that time, it was 6 feet deep."

$$\overline{3} \times \overline{2} = 6$$

"If $\overline{3}$ suggests a trip of 3 miles west, then $\overline{3} \times 2$ would suggest a trip 6 miles west. But if $\overline{3}$ times 'two' suggests a trip 6 miles west, then 3 times 'the opposite of two' could suggest a trip 'the opposite of 6 miles west'—or 6 miles east."

Don't struggle to find such examples—they are hardly convincing. All we can do is realize that there is nothing necessarily inconsistent here. If you want, you can add the following argument:

$$\overline{3} \times (\overline{2}+2) = \_\_\_\_$$

Since $\overline{2} + 2 = 0$, and 0 is either of a pair of factors, the product is 0—then, $\overline{3} \times 0 = 0$.

But consider it differently. We certainly want Ben's "Do-It-to-Both" rule to hold—the "distributive property," as mathematicians call it. So, we want it to be the case that:

$$\overline{3} \times (\overline{2}+2) = (\overline{3} \times \overline{2}) + (\overline{3} \times 2)$$

Since that first expression is another name for 0, the other must also be a name for 0. Therefore:

$$(\overline{3} \times \overline{2}) + (\overline{3} \times 2) = 0$$

If we have already agreed that $\overline{3} \times 2 = 6$, then we can rewrite that last expression as:

$$(\overline{3} \times \overline{2}) + (\overline{6}) = 0$$

How can we make the following statement true?

$$\_\_\_\_ + \overline{6} = 0$$

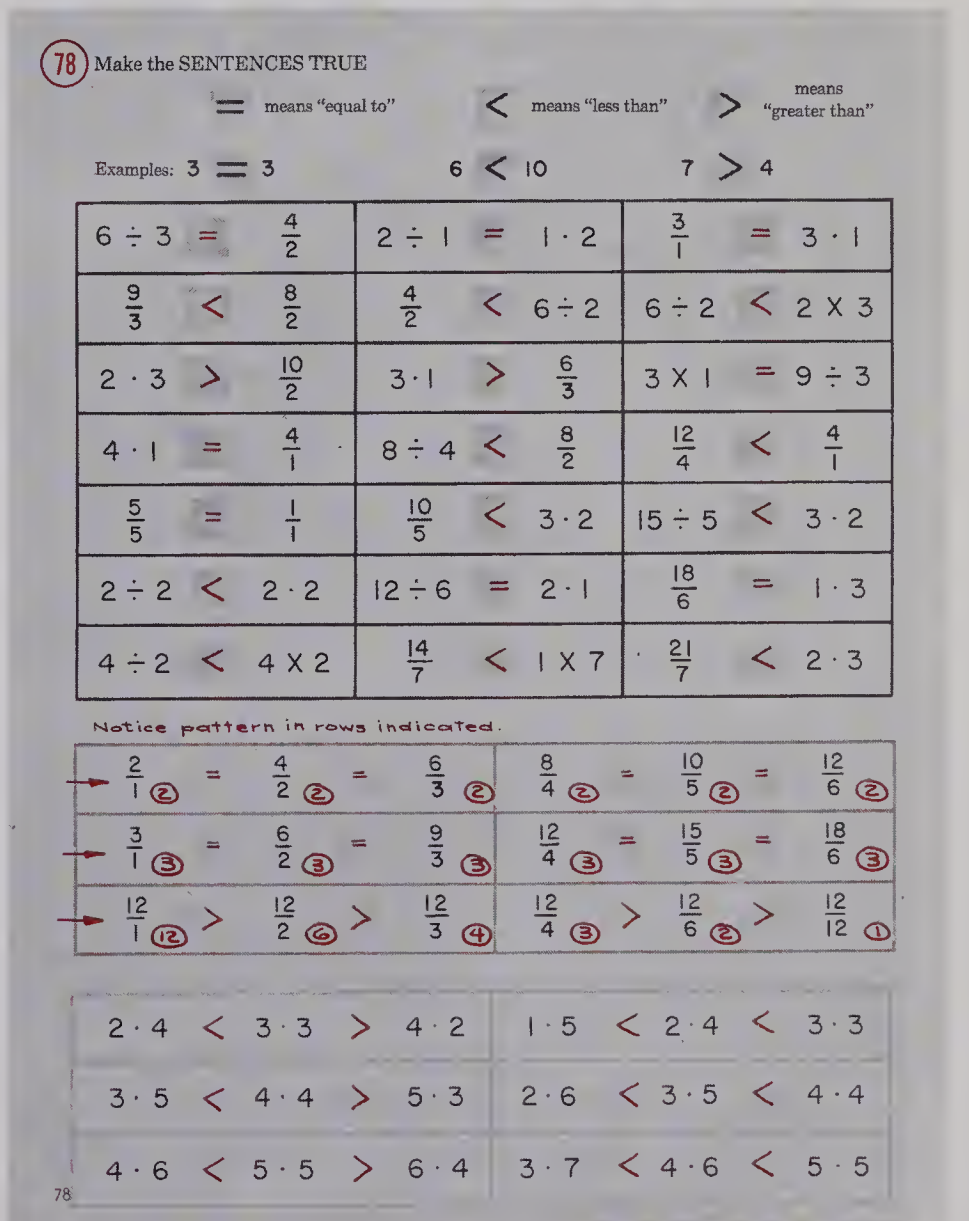
The only was is to note 6 as the other addend, so it reads $6 + \overline{6} = 0$. Then it must follow that:

$$\overline{3} \times \overline{2} = 6$$

(But don't call this a "proof." It resembles one, but it is only an argument that seems to strengthen our belief that these results may be consistent with the rest of our results.)

Notes to accompany *Work Text* page 78

Children often forget that in arithmetic they are continually reading and writing in shorthand. The symbols



$7 + 3 = 10$

are shorthand for a statement: Seven plus three equals ten. Usually these statements make an assertion. Certainly the example is an assertion.

Here is a sentence that is not taken from mathematics: "John swam the river."

We recognize it as a well-constructed sentence: it has a subject and predicate; the verb is an acceptable from of the infinitive, "to swim"; we are satisfied that it is grammatically correct. We all agree that it is a perfectly good sentence.

Is this a perfectly good mathematical sentence: $7 + 3 = 29$? The structure of the sentence is entirely acceptable. The shorthand says, "seven plus three equals twenty-nine;" and that is a well-constructed sentence.

We are disturbed, however. The sentence is an assertion; it is a complete sentence; it is all very fine, except the assertion made is false.

It may also be a fact that John cannot swim. However, we were not disturbed by the assertion that "John swam the river," because we had no way of knowing it could not be true. We are disturbed by an assertion that $7 + 3 = 29$ because, while it is a well constructed sentence, it simply isn't true. It is a false assertion, a false sentence, a statement of a falsehood.

Mathematical statements may be false

In mathematics, we often find assertions that are false. Sometimes, they arise from having made a mistake. Sometimes, the only way to prove that a proposition is false is to accept it tentatively and follow out its consequences until we reach a conclusion that is obviously false and, at that point, we say we have disproved the proposition. (Later, we will consider the statement that $3/0 = 3$ is an improper statement by considering the consequence. If it is true, then $3 = 3 \times 0$, but that is false. Therefore, the first statement is somehow improper.)

Children should have wide experience in deciding whether mathematical sentences are true or false. We do them a disservice when we work diligently to protect them from seeing a false assertion.

One of the three signs of relationship are available to complete the statements and make them true: $=$, $<$, and $>$.

When *Work Text* page 78 has been completed, some pupils may have noticed, in the bottom group, the patterns in the rows and in the blue section in the columns. This page serves as a check up on ability to handle content developed on earlier pages.

Reteaching the page

Another approach to this page is to consider each problem in the context of another form:

			=	≠	<	>
$6 \div 3$	□	$\frac{4}{2}$	T	F	F	F
$\frac{9}{3}$	□	$\frac{8}{2}$	F	T	T	F
$2 \cdot 3$	□	$\frac{10}{2}$				

This form asks essentially the same questions as page 78, but in a slightly different way. It introduces the sign "≠," which stands for " . . . is not equal to . . . " The child is asked to imagine each of the four basic signs of equality and inequality in the shaded box, and to determine if the statement is true or false. He marks F or T in each of the four columns. Again, some apparently difficult questions can be answered by argument rather than computation. Such argument should be encouraged—and computation discouraged. Plenty of practice will be provided in which computation is required.

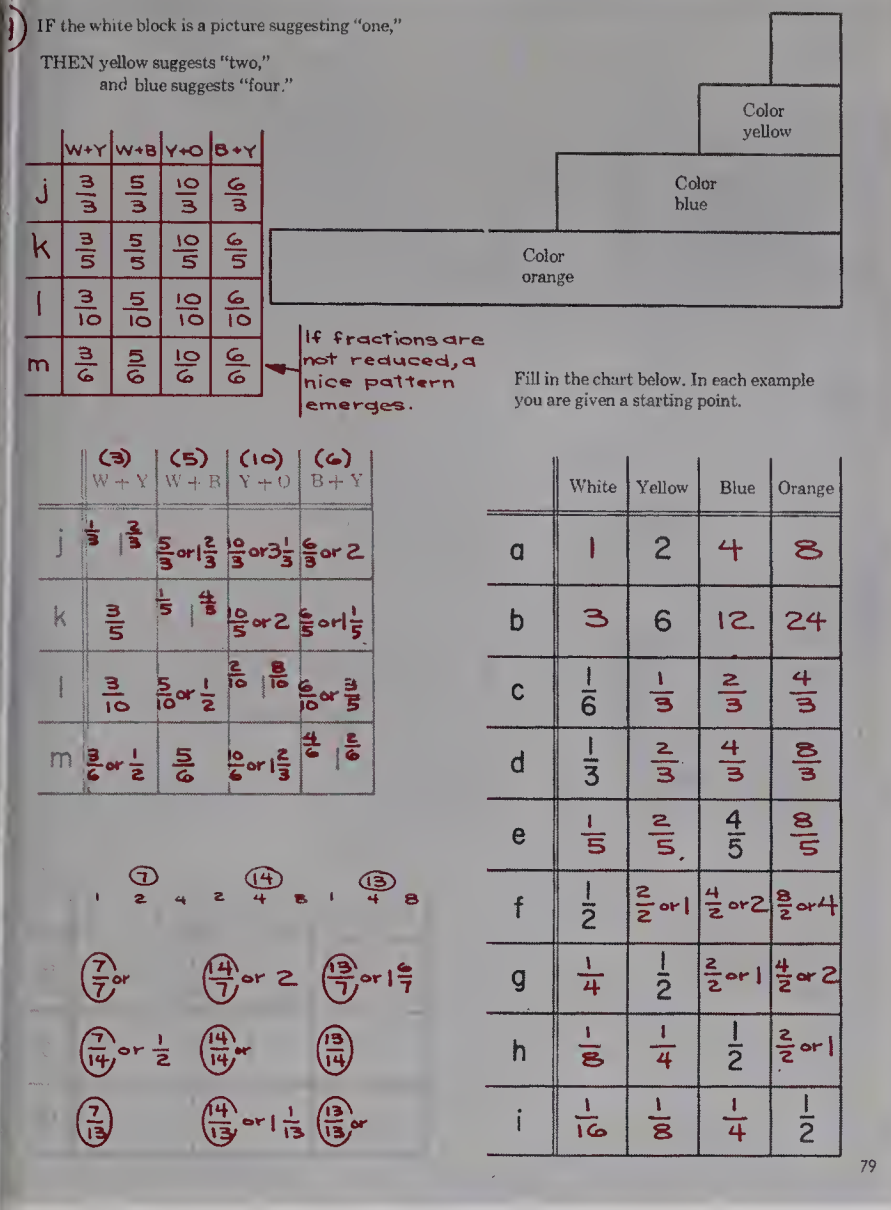
After the page has been completed, the answers will reveal a pattern well worth discussing. Wherever there is a T in the first column, the rest of the row is F. Wherever an F appears in the first column, then the rest of the row includes two T's and one F. Why are there never three T's or three F's in the rest of the row? Why is the answer in the first column always the opposite of the answer in the second column—under ≠?

Notes to accompany *Work Text* page 79

An oft-repeated sentence in mathematics is in the form of an assertion that, if something we point out is true, it follows as a consequence that something else we note is true.

If _____,
then _____.

"*If* it is agreed that the area of a small square suggests 1, *then* the area of the rectangle to be colored yellow on page 79 will suggest 2 . . . because the rectangle is twice as large as the small square. Further, it follows that the rectangle to be colored blue will suggest 4, and the rectangle to be colored orange will suggest 8."


IF the white block is a picture suggesting "one,"
THEN yellow suggests "two,"
and blue suggests "four."
Color yellow
Color blue
Color orange
If fractions are not reduced, a nice pattern emerges.
Fill in the chart below. In each example you are given a starting point.
79

	White	Yellow	Blue	Orange
a	1	2	4	8
b	3	6	12	24
c	1/6	1/3	2/3	4/3
d	1/3	2/3	4/3	8/3
e	1/5	2/5	4/5	8/5
f	1/2	2/2 or 1	4/2 or 2	8/2 or 4
g	1/4	1/2	2/2 or 1	4/2 or 2
h	1/8	1/4	1/2	2/2 or 1
i	1/16	1/8	1/4	1/2

In the chart, at line *a*, this same idea is brought out by asking the question: "If the area of the yellow shape suggests 2, then the area of the white shape would suggest 1, the blue area would suggest 4, and the orange area would suggest 8." These results should be recorded after *a*.

The given information in line *b* asks a new question: "If the size of the yellow area suggests 6, then what does the size of the other areas suggest?"

Each line of the chart asks a different question: What are the consequences of the information supplied in each? "If white suggests 1/6, then. . . ."

Examples *j*, *k*, *l* and *m* ask similar, but more difficult, questions. Example *j* asks: "If the size of the combined area of the white and yellow blocks suggests 1, then what does the size of the combined areas of white and blue suggest; of yellow and orange; of blue and yellow?"

Since white and yellow together suggest 1, the white must suggest 1/3, the yellow must suggest 2/3, the blue must suggest 1 1/3, and the orange must suggest 2 2/3. These results might be more obvious if written as common fractions:

1/3, 2/3, 4/3, 8/3

From this, we find that white and blue together must, based on the information given in row *j*, suggest 1/3 + 4/3 or 5/3 or 1 2/3; yellow and orange together must suggest 2/3 + 8/3 or 10/3 or 3 1/3; and blue and yellow together must suggest 2/3 + 4/3 or 6/3 or 2.

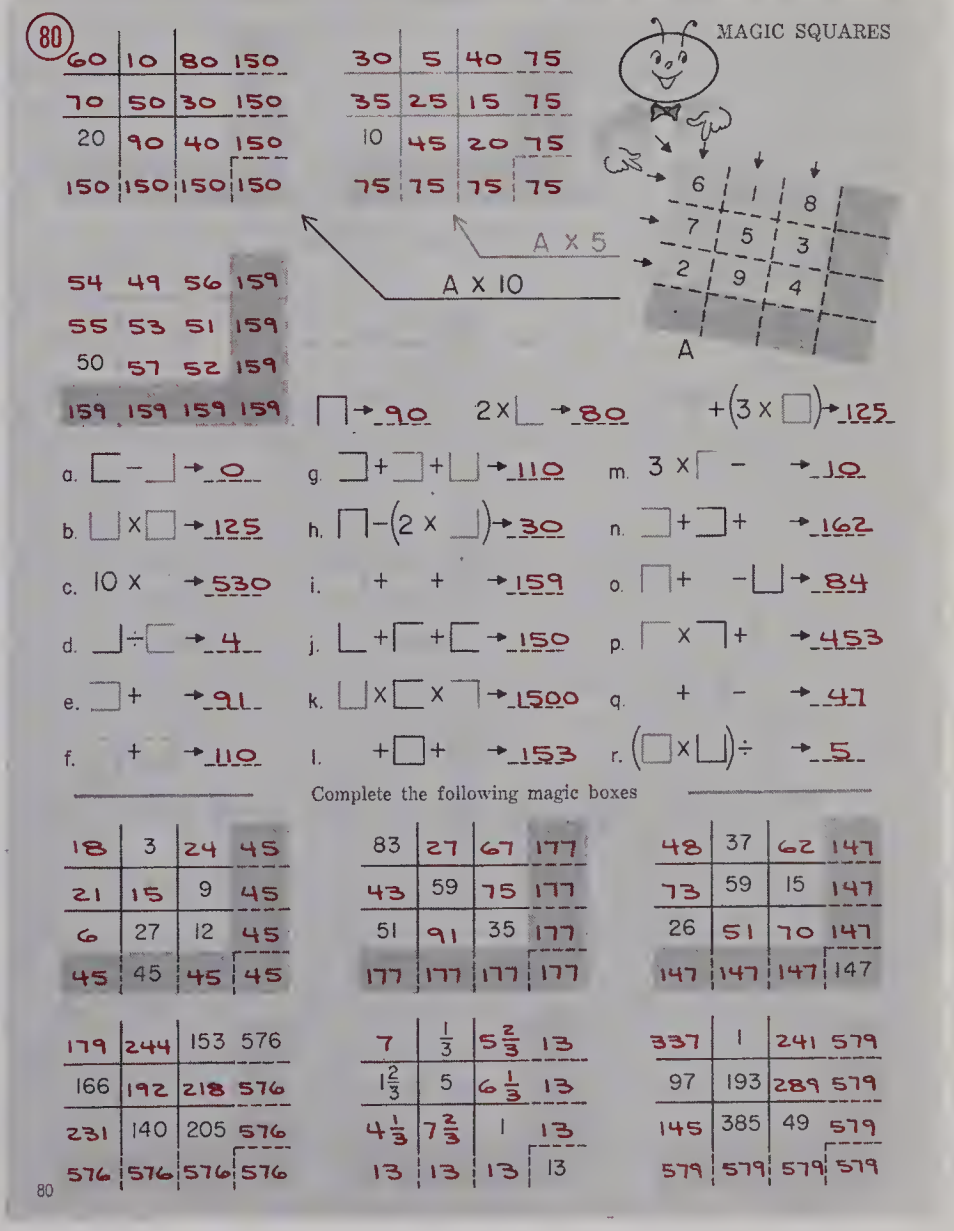
Examples *k*, *l*, and *m* depend on similar reasoning. Examples *n*, *o* and *p* are even more difficult. In example *n*, the information given is that if the size of the white, yellow and blue areas taken together suggest the idea of 1, then what about the other combinations listed?

It would take 7 of the white squares to cover the white, yellow and blue. If the total area suggests 1, then white must suggest 1/7, yellow must suggest 2/7, and blue must suggest 4/7—so orange must suggest 8/7. Then the combination of yellow, blue and orange must point to 2/7 + 4/7 + 8/7 or 14/7 or 2.

The other examples will yield to the same kind of analysis.

Notes to accompany *Work Text* page 80

Page 80 depends on the Magic Square in the upper right-hand corner. From it, three other Magic Squares are generated by applying the rules indi-



cated, along with the arrows. The numbers noted in the square of green lines are 5 times the numbers written in the corresponding cells of the initiating square.

"Is the new square magic? Is the sum of each column, each row, and each main diagonal the same number?"

In each cell in the square with black lines, a number is noted that is 10 times the number noted in the corresponding cell of the first square. "Add 48" is the rule which, applied to the numbers of the original group, produces those noted in the grid of blue lines.

Numbers for the expressions in each of the examples are indicated by the shape and color in each instance.

In the three examples at the top, the first is obviously found in the middle of the bottom row of the square of black lines – 90. The second calls for twice the number noted in the upper right hand corner of the square of green lines – $2 \times 40 = 80$. The third calls for the sum of the number written in the bottom left hand corner of the square of blue lines and three times the number in the center of the square of green lines – $50 + (3 \times 25) = 50 + 75 = 125$.

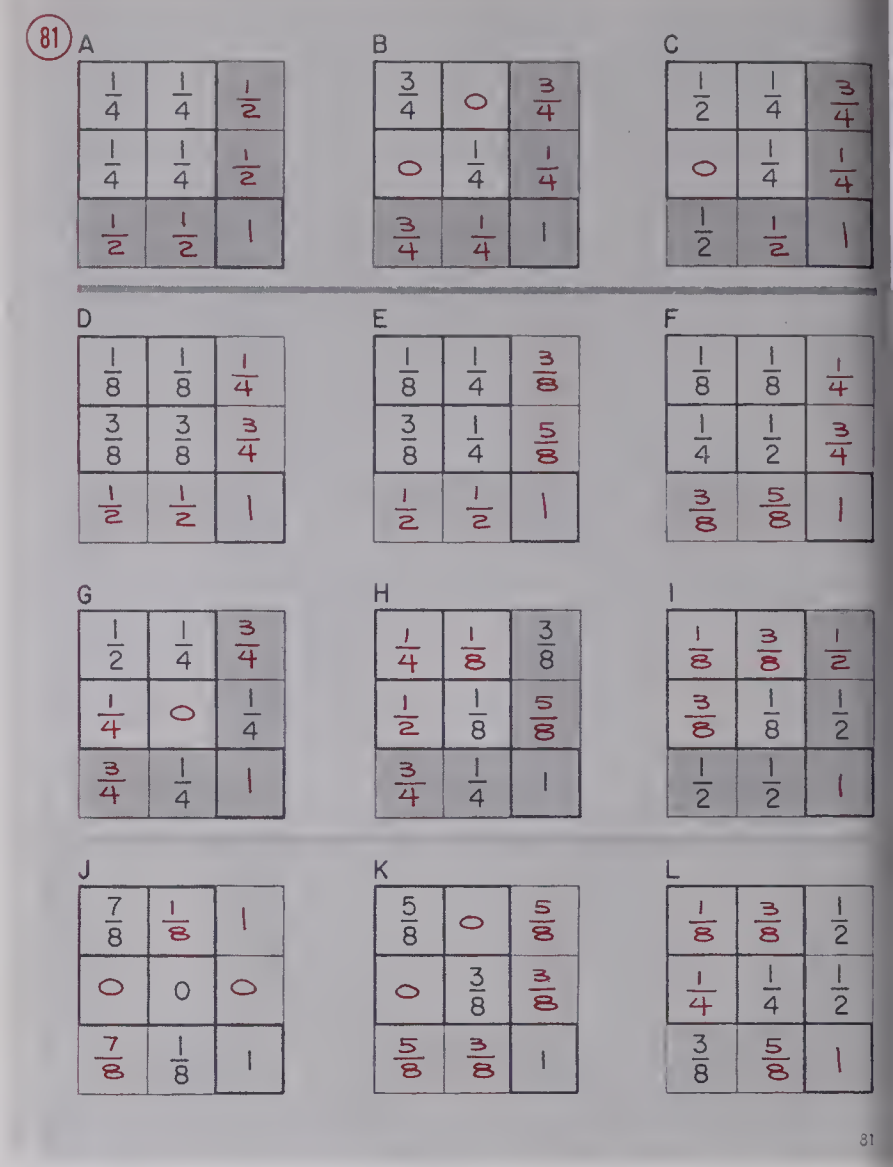
In the six unfinished examples at the bottom of the page, it is assumed that each is to have the "magic" quality when completed – the sum of the numbers in each row, diagonal and column name the same number. Enough information is provided in each so that there is only one way it can be completed.

In the top left example, the row-column-diagonal sum is indicated as 45. The sum of 3 and 27 is 30. So, the missing cell must contain 45 - 30 or 15.

Two numbers are given for the right-hand column. Thus ____ + 9 + 12 = 45, or ____ + 21 = 45, or 45 - 21 = 24. By similar reasoning and computations, all six squares can be reconstructed so that they preserve the magic quality.

Notes to accompany *Work Text* page 81

Page 81 is a refresher on the subject of addition (and subtraction) of common fractions. The rule is "addition." While A through F are concerned only with addition, the rest involve examples of subtraction along the way. (It may come as a surprise to some that all examples on this page lead to the same number being written in all check boxes: 1.)



It might be argued that insufficient information is provided in K. However, the fractions given in the basic four cells are 5/8 and 3/8. Their sum is 1 – and the check box indicates that the sum of numbers written in all four cells is 1. Therefore, the empty cells can only contain 0 (unless someone wants to use some other number and its opposite).

If there is too much hesitation on any of the examples, extended review can be built around a similar activity using some of the Cross Number Puzzle blanks in the supplemental *Work Forms*.

Notes to accompany *Work Text* pages 82-85

The division games on pages 82 through 85 focus the spotlight on the meaning in this operation. To understand the steps of long division, rather than to have memorized them, requires a realization of three important ideas which have been developing continuously in the *Math Workshop* program in a number of different contexts:

1) A number can be regrouped. Here are some examples:

$30 \rightarrow 20 + 10$
$30 \rightarrow 18 + 12$
$30 \rightarrow 16 + 8 + 6$
$30 \rightarrow 12 + 27 - 9$

2) Division is the inverse of multiplication:

$8/2 = 4$ because $2 \times 4 = 8$

$4\overline{)12}$ (3) because $3 \times 4 = 12$

$10 \div 2 = 5$ because $5 \times 2 = 10$

3) In multiplication (and division) you can "Do-It-to-Both." If you want the product of the factors 5×8, you may first wish to regroup one or the other of the factors before multiplying, and add the partial products. Some ways of arriving at an answer to the problem are:

(1) $5 \times 8 = 40$
(2) $5 \times (2+6) = 5 \times 2 + 5 \times 6 = 10 + 30 = 40$
(3) $5 \times (3+5) = 5 \times 3 + 5 \times 5 = 15 + 25 = 40$
(4) $(3+2) \times 8 = 3 \times 8 + 2 \times 8 = 24 + 16 = 40$
(5) $(2+3) \times (4+4) = 2 \times 4 + 2 \times 4 + 3 \times 4 + 3 \times 4 = 8 + 8 + 12 + 12 = 40$

Since division is the inverse of multiplication, you may first wish to regroup the product in a division problem and "Do-It-to-Both." So:

(1) $40/8 = 5$
(2) $40/8 = \dfrac{16 + 24}{8} = 2 + 3 = 5$
(3) $40/8 = \dfrac{8 + 32}{8} = 1 + 4 = 5$

or $8\overline{)8 + 32}$ $1 + 4 = 5$

(4) $40/8 = 5 + 2 - 2 = 5$

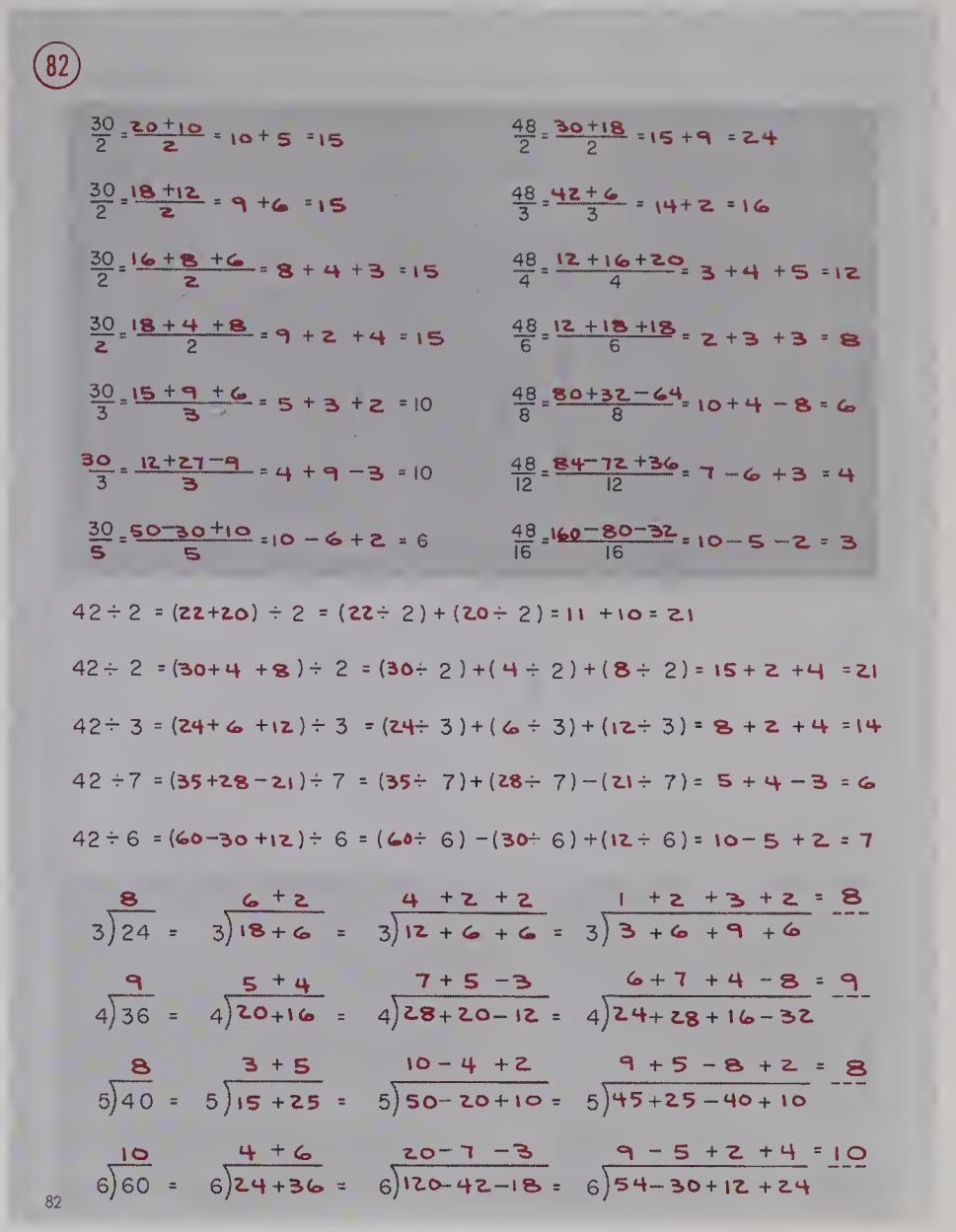
or $8\overline{)40 + 16 - 16}$ $5 + 2 - 2 = 5$

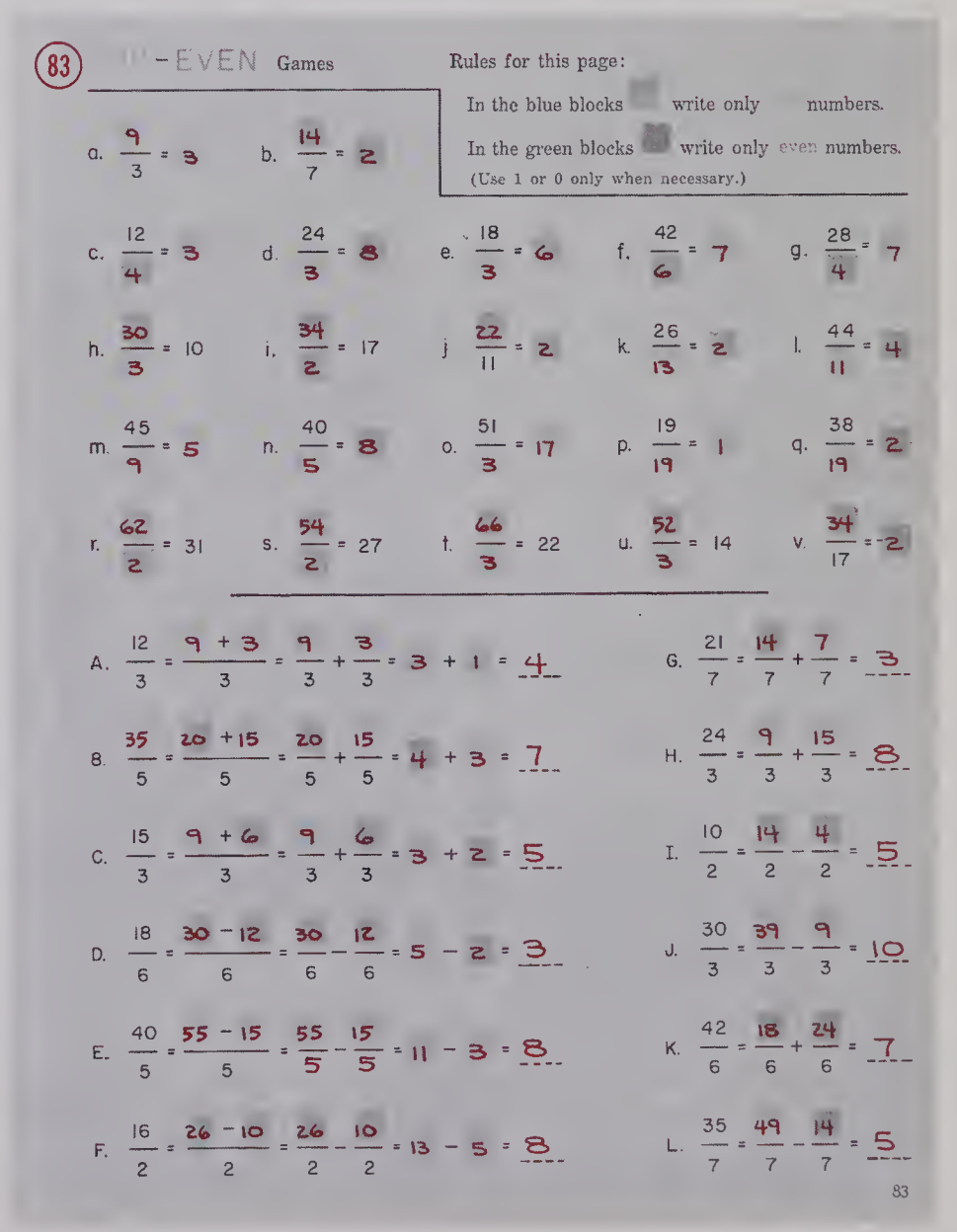
A no-readiness approach

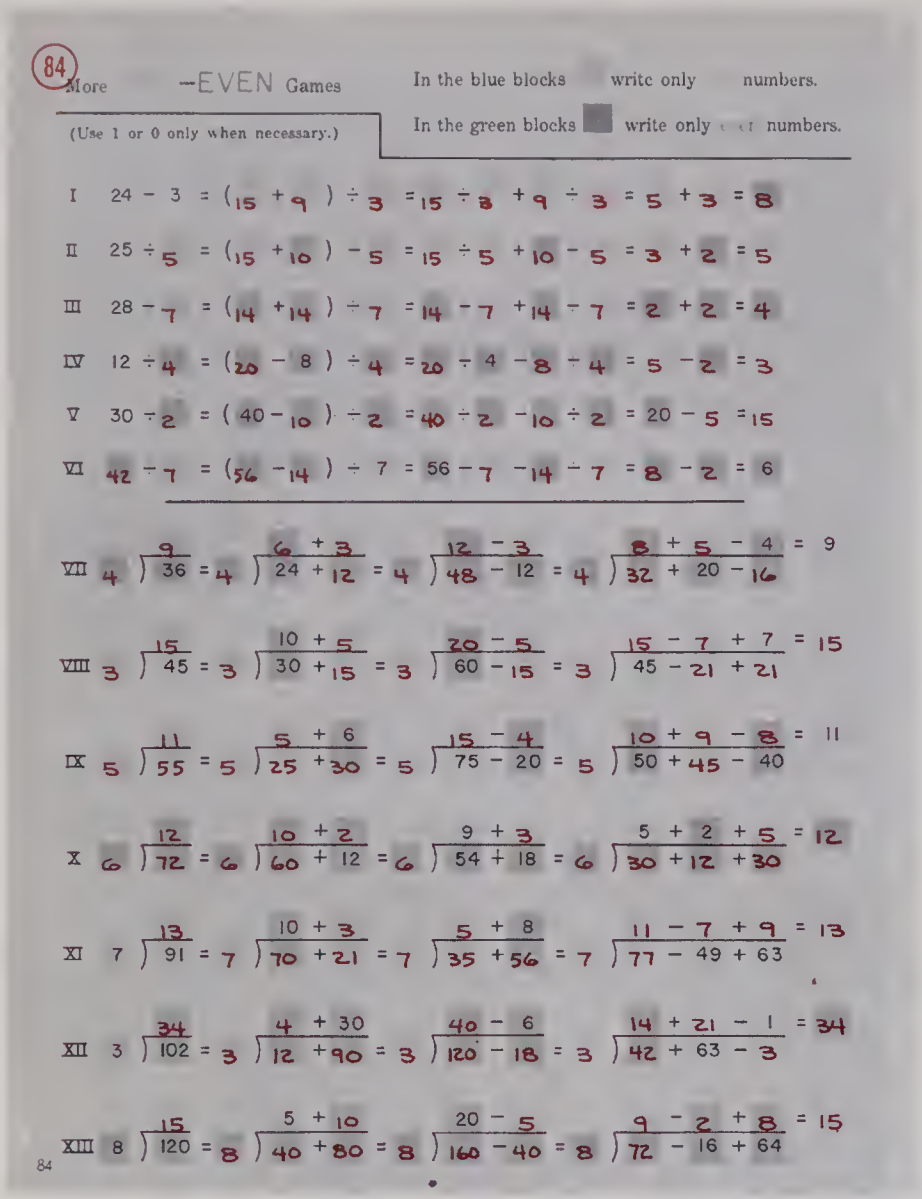
Try introducing these pages without preparing pupils. Such a "no-readiness" approach should be used more often than specifically indicated in this guide. Whenever you feel the children are ready to tackle any page without explanation, try it. You can always interrupt and insert whatever class discussion and work you feel is needed.

Another very effective way to handle certain pages is to divide the class into small groups that will work together. Let each group talk about the page. Urge them to do a little mind reading.

"What work do you think needs to be done on this







84 More —EVEN Games

In the blue blocks write only numbers.
In the green blocks write only numbers.

(Use 1 or 0 only when necessary.)

I $24 - 3 = (15 + 9) \div 3 = 15 \div 3 + 9 \div 3 = 5 + 3 = 8$

II $25 \div 5 = (15 + 10) - 5 = 15 \div 5 + 10 - 5 = 3 + 2 = 5$

III $28 - 7 = (14 + 14) \div 7 = 14 - 7 + 14 - 7 = 2 + 2 = 4$

IV $12 \div 4 = (20 - 8) \div 4 = 20 \div 4 - 8 \div 4 = 5 - 2 = 3$

V $30 \div 2 = (40 - 10) \div 2 = 40 \div 2 - 10 \div 2 = 20 - 5 = 15$

VI $42 \div 7 = (56 - 14) \div 7 = 56 - 7 - 14 \div 7 = 8 - 2 = 6$

VII $4\overline{)36} = 4\overline{)24 + 12} = 4\overline{)48 - 12} = 4\overline{)32 + 20 - 16} = 9$

VIII $3\overline{)45} = 3\overline{)30 + 15} = 3\overline{)60 - 15} = 3\overline{)45 - 21 + 21} = 15$

IX $5\overline{)55} = 5\overline{)25 + 30} = 5\overline{)75 - 20} = 5\overline{)50 + 45 - 40} = 11$

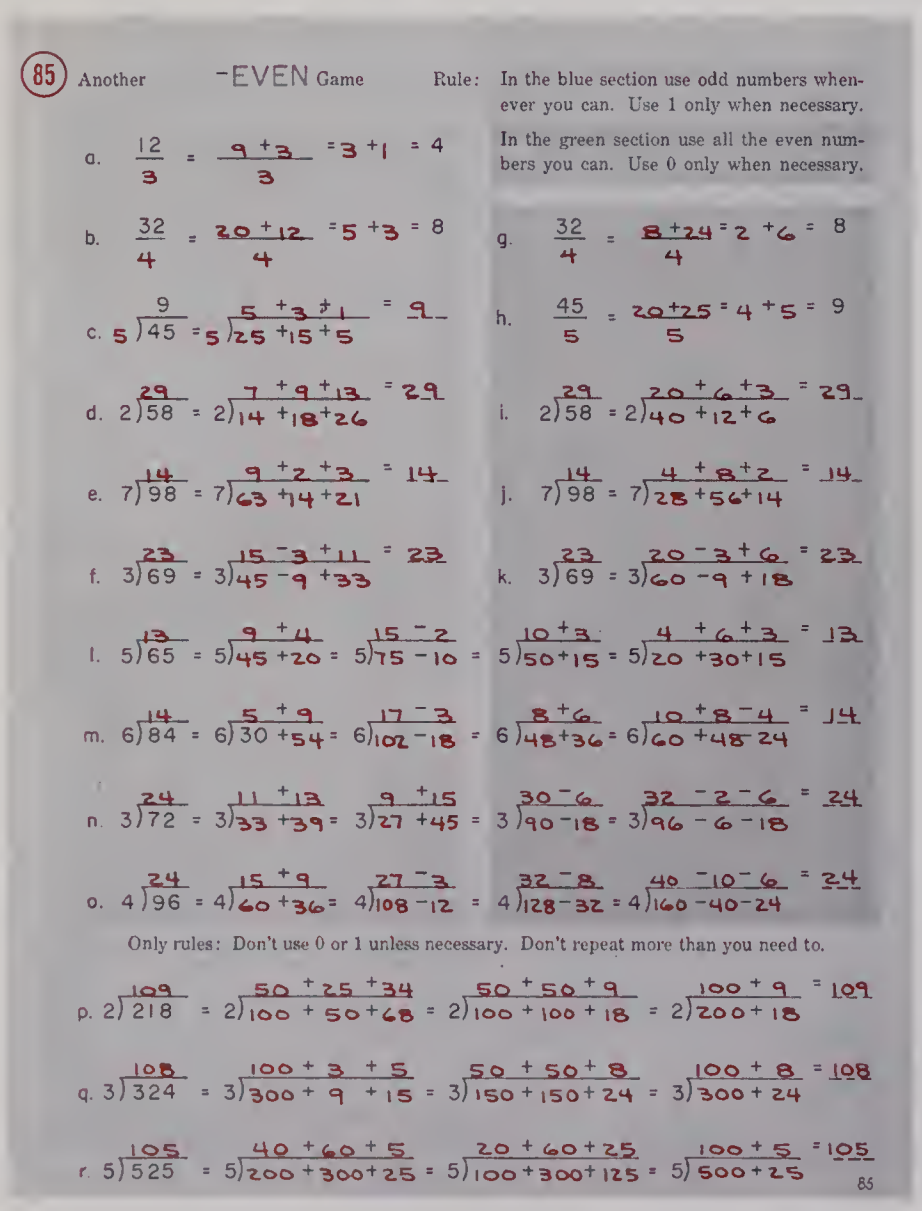
X $6\overline{)72} = 6\overline{)60 + 12} = 6\overline{)54 + 18} = 6\overline{)30 + 12 + 30} = 12$

XI $7\overline{)91} = 7\overline{)70 + 21} = 7\overline{)35 + 56} = 7\overline{)77 - 49 + 63} = 13$

XII $3\overline{)102} = 3\overline{)12 + 90} = 3\overline{)120 - 18} = 3\overline{)42 + 63 - 3} = 34$

XIII $8\overline{)120} = 8\overline{)40 + 80} = 8\overline{)160 - 40} = 8\overline{)72 - 16 + 64} = 15$

84



85 Another —EVEN Game

Rule: In the blue section use odd numbers whenever you can. Use 1 only when necessary.

In the green section use all the even numbers you can. Use 0 only when necessary.

a. $\frac{12}{3} = \frac{9+3}{3} = 3+1 = 4$

b. $\frac{32}{4} = \frac{20+12}{4} = 5+3 = 8$

c. $5\overline{)45} = 5\overline{)25+15+5} = 9$

d. $2\overline{)58} = 2\overline{)14+18+26} = 29$

e. $7\overline{)98} = 7\overline{)63+14+21} = 14$

f. $3\overline{)69} = 3\overline{)45-9+33} = 23$

g. $\frac{32}{4} = \frac{8+24}{4} = 2+6 = 8$

h. $\frac{45}{5} = \frac{20+25}{5} = 4+5 = 9$

i. $2\overline{)58} = 2\overline{)40+12+6} = 29$

j. $7\overline{)98} = 7\overline{)28+56+14} = 14$

k. $3\overline{)69} = 3\overline{)60-9+18} = 23$

l. $5\overline{)65} = 5\overline{)45+20} = 5\overline{)75-10} = 5\overline{)50+15} = 5\overline{)20+30+15} = 13$

m. $6\overline{)84} = 6\overline{)30+54} = 6\overline{)102-18} = 6\overline{)48+36} = 6\overline{)60+48-24} = 14$

n. $3\overline{)72} = 3\overline{)33+39} = 3\overline{)27+45} = 3\overline{)90-18} = 3\overline{)96-6-18} = 24$

o. $4\overline{)96} = 4\overline{)60+36} = 4\overline{)108-12} = 4\overline{)128-32} = 4\overline{)160-40-24} = 24$

Only rules: Don't use 0 or 1 unless necessary. Don't repeat more than you need to.

p. $2\overline{)218} = 2\overline{)100+50+68} = 2\overline{)100+100+18} = 2\overline{)200+18} = 109$

q. $3\overline{)324} = 3\overline{)300+9+15} = 3\overline{)150+150+24} = 3\overline{)300+24} = 108$

r. $5\overline{)525} = 5\overline{)200+300+25} = 5\overline{)100+300+125} = 5\overline{)500+25} = 105$

85

page? Talk about it together. If you seem to agree, go ahead. I am ready to help if you are unable to get started on your own. Help each other as much as you can."

This introduction to a new page has a note of challenge in it . . . and just a little flavor of the puzzle. These are two essential ingredients in creative mathematics.

In mathematics, as in other subject matter areas, there is a long tradition of spoon-feeding each prescribed dose, always pointing out clearly what is to be done next. Children become dependent on directions, instructions, orders of procedure.

We need to create every opportunity we can to urge children to consider a problem that is, at first, not clear. Where does one begin? Is there a part of the problem that looks familiar? If so, can this be a clue to the less familiar areas? What's going on?

Almost all pages in the *Work Text* will give up their secrets to a group of children anxious to prove their ability. Often, a child working alone gets lost in a side road that does not help move toward the goal. But this is seldom the case with a small group.

It is essential that the chances of success be good. Be quite sure that your class is prepared for pages you ask them to tackle on their own. Within a few minutes, you can easily tell whether they are. If they are not, give them help and return to the battle another day.

Anyone can ask an impossibly difficult question or set an impossible task; anyone can ask a trivial question that everyone can answer without effort; but the only good questions are those that challenge the student and yield their answers in response to sustained effort—and these are hard questions to formulate.

The pages children tackle on their own should be neither too hard nor too easy—the pages should challenge the class to think their way through to a successful conclusion.

Extending the activity

Three important extensions of the work on these pages should involve:

1. Writing the mathematical statements in various forms;
2. Drawing pictures; and
3. Writing word stories that are suggested by the mathematical statements.

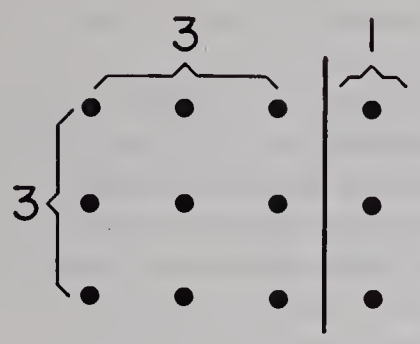
For example:

1. $3\overline{)12} = 3\overline{)9+3}$ $\quad 3+1=4$

$$12/3 = \frac{9+3}{3} = 9/3 + 3/3 = 3 + 1 = 4$$

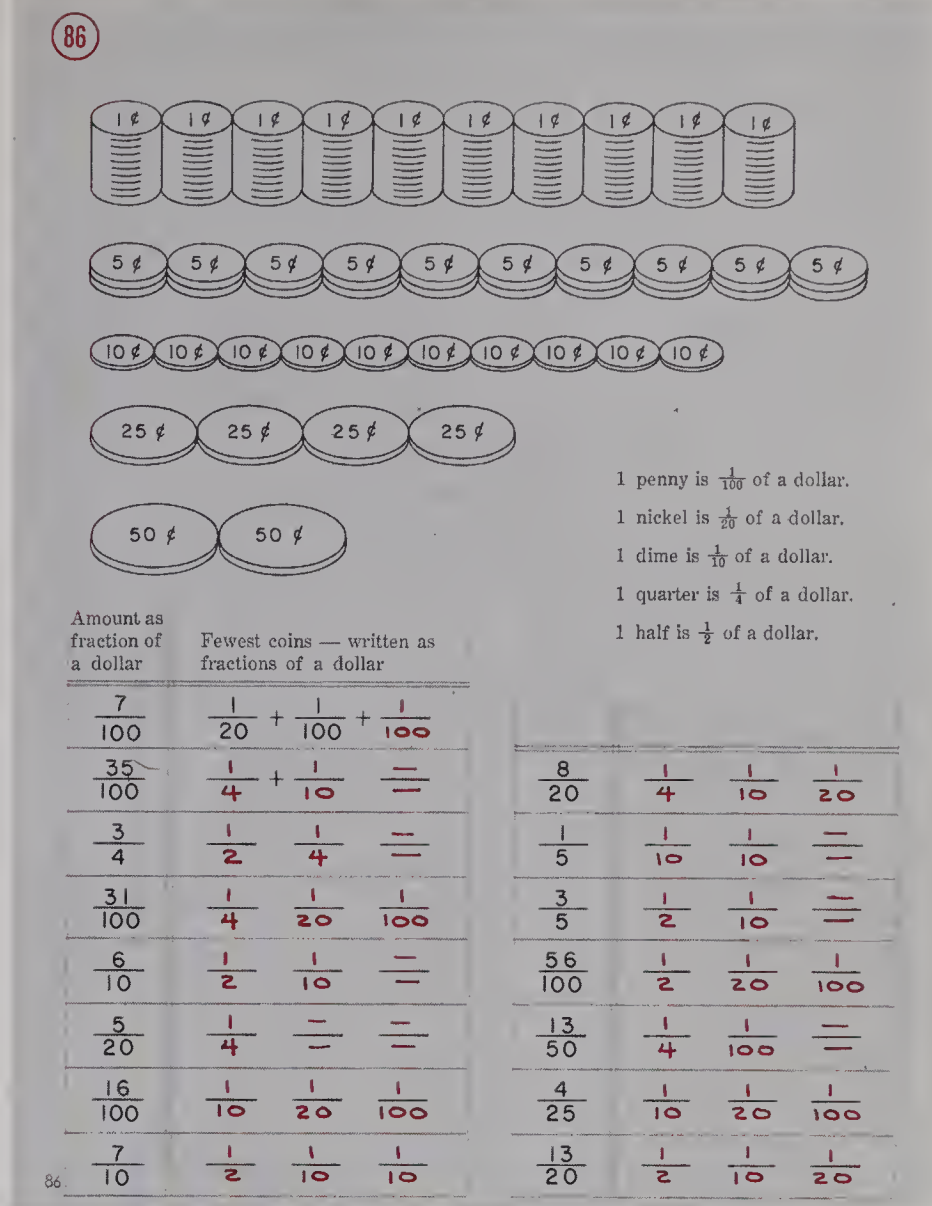
$$12 \div 3 = (9 \div 3) + (3 \div 3) = 3 + 1 = 4$$

2.



3. I have to divide 12 pieces of fruit among 3 children. If I have 9 apples and 3 pears, what would be a good way to do it?

Teams may work on the problem of creating word problems with the class voting on the most creative or amusing examples.



Notes to accompany *Work Text* page 86

Page 86 is based on a version of "fewest coins." A fractional amount of a dollar is given. This fraction is to be written as the sum of fractions, each fraction used indicating the coins required to make up the indicated amount. The first example refer to 7/100 of a dollar, or 7¢. This can be made up with fewest coins by one nickel and two pennies. This can be suggested by the expression:

$$7/100 = 1/20 + 1/100 + 1/100$$

If some would prefer the form

$$7/100 = 1/20 + 2/100$$

they should not be told they are wrong. If the first form is to be preferred, then a rule has to be made for this game that each fraction used must suggest a single coin. Thus, the number of addends would indicate the "number of coins"—the fact in which we are interested.

Another version of this game can be played with much the same rule. The initial amount may be written as a sum of fractions—to be rewritten to indicate the fewest coins.

$3/4 + 4/20 = 1/2 + 1/4 + 1/10 + 1/10$
$3/8 + 1/8 = 1/2$
$2/3 + 1/12 = 1/2 + 1/4$
$1/2 - 1/20 = 1/4 + 1/10 + 1/10$
$1/4 + 1/10 + 1/4 - 1/2 = 1/10$
$4/20 - 1/5 + 7/100 = 1/20 + 1/100 + 1/100$

This may be used for a competition between two teams, each taking turns in challenging the other to rewite an indicated part of a dollar in "fewest coins."

Notes to accompany *Work Text* pages 87-90

Pages 87, 88, 89 and 90 introduce the decimal notation as shorthand for fractions such as those used in the game on page 86. You may be surprised that this is done without any discussion of place value in those terms.

Comments on place value

Very early in organized number experiences, children count easily beyond 9 to 10, 11, 12, etc. They are not aware that a most important idea was revealed in the creation of the numeral 10 . . . it is composed of two previously used numerals. Nor does it dawn on them for a long time that they keep on dealing with numerals that are combinations of the first ten numerals 0 through 9.

Words	Three kinds of shorthand A	B	C	In C, write twice as much	half as much
one dollar	$1	100¢	$1.00	$2.00	$.50
one dime	$\frac{1}{10}$	10¢	$.10	$.20	$.05
one penny	$\frac{1}{100}$	1¢	$.01	$.02	$.00½
one and one-half dollars	$1\frac{1}{2}$	150¢	$1.50	$3.00	$.75
fifteen dimes	$\frac{15}{10}$ or $1\frac{5}{10}$	150¢	$1.50	$3.00	$.75
twenty-four cents	$\frac{24}{100}$	24¢	$.24	$.48	$.12
two dollars, forty-eight cents	$2\frac{48}{100}$	248¢	$2.48	$4.96	$1.24
one hundred eighty-four cents	$\frac{184}{100}$	184¢	$1.84	$3.68	$.92
one dollar and a quarter	$1\frac{1}{4}$	125¢	$1.25	$2.50	$.62½
three dollars seventy-five cents	$3\frac{75}{100}$	375¢	$3.75	$7.50	$1.87½
eight hundred twenty-two cents	$\frac{822}{100}$	822¢	$8.22	$16.44	$4.11

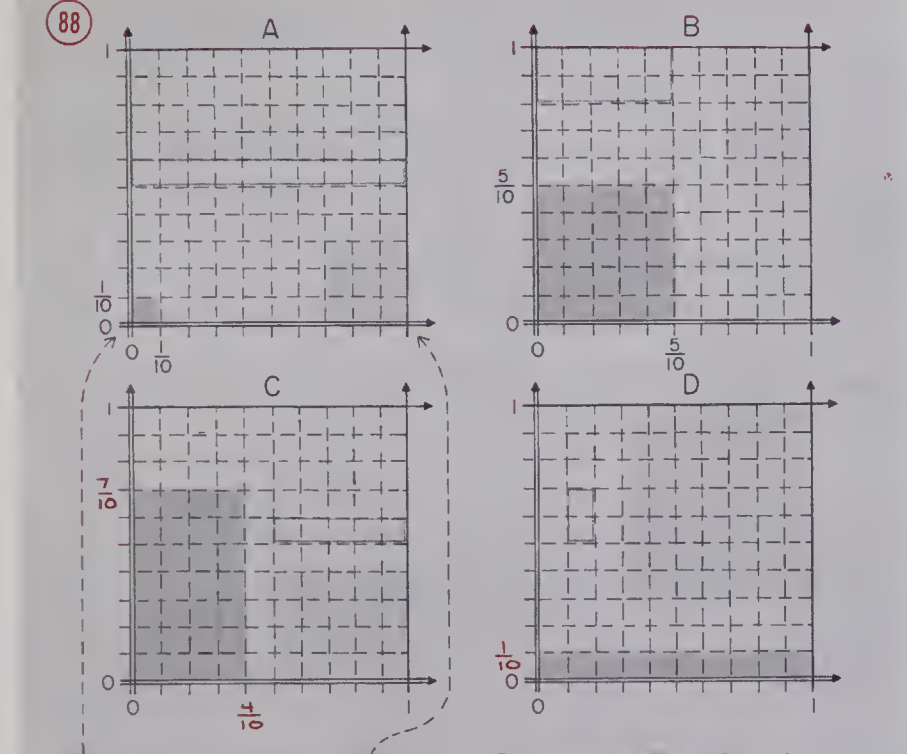
Mary, Alice, and Jane operated a refreshment stand one summer.

Each day, they paid for everything they sold. The money they had left was profit.

Each day, they divided the profits evenly among themselves.

Below are partial records of one week's activity. Can you complete the record?

	Profit in numbers of			Profit in number of fewest coins			Each child's share in number of		
	Dollars	Dimes	Pennies	Dollars	Dimes	Pennies	Dollars	Dimes	Pennies
Mon.	1	7	22	1	9	2	0	6	4	0
Tues.	2	11	16	3	2	6	1	0	8	2
Wed.	1	18	36	3	1	6	1	0	5	1
Thurs.	0	21	41	2	5	1	—	8	3	2
Fri.	1	13	88	3	1	8	1	0	6	0
Sat.	0	27	58	3	2	8	1	0	9	1
Totals	5	97	261	17	3	1	5	7	7	0



This ingenious invention—out decimal system of numeration—is, of course, a fundamental idea of arithmetic. In the past, the importance of this fact has led to much early talk about "place value." There seemed to be a hurry to introduce abstract terms in order to make it possible to talk about place value. Some educators were anxious to get to "carrying" and "borrowing," and hastened to point out the idea of place value as one of the main features of our modern system of numeration.

The *Math Workshop for Children* has consistently introduced each use of the idea of place value and its extension (to decimal notation in the present instance) through discovery. We have carefully withheld any names for ideas—abstractions—until those ideas are clearly understood by children on a sub-vocal or non-verbal level.

Grasp of an idea

"How can you tell when an idea has been grasped unless you talk about it?" This question might be answered by the assertion that if you talk about an idea, you can *never* be sure the child understands it . . . at least when it is a new idea. As soon as talk begins, some children begin parroting terms and expressions, rather than expressing ideas.

If you want to know whether or not a child has grasped the idea, give him some examples in which he can use the idea. Then watch him proceed. Without using words, he will reveal his facility with the idea.

For example, does a child in the first grade have trouble counting past 9, writing numerals past 9, or finding the following sums?

2 + 9 = ____	10 + 10 = ____	18 + 4 = ____
5 + 10 = ____	20 + 15 = ____	14 + 18 = ____

If he has difficulty in reaching a conclusion in each case, then he has not grasped the idea of addition—or he is reticent about putting his understanding to work.

One way he can approach each example is to count out two piles of objects, with the indicated number in each. Now, by counting both piles separately, he can arrive at the sum. If he cannot, he either does not know how to count or he does not understand addition. In which case, it would be woefully premature to talk about "place value" or "regrouping."

Another way he might find, say, the sum of 18 and 4 would be to start at 18 and count the next four numbers—19, 20, 21 and, finally, 22—18 + 4 = 22.

Beyond these methods, he might regroup in many ways. He might argue that 18 = 12 + 6 and then look at the problem this way:

$$18+4 = 12+(6+4) = 12+10 = (12+8) + 2 = 20+2$$

or some other method of regrouping.

Regrouping

Unless he knows that there are an unlimited number of ways in which he can regroup, he does not understand regrouping. If he does understand this almost unlimited freedom, then he will discover the convenient and time saving ways to regroup. There need be no discussion at all about place value.

Premature discussion of place value gives the child a narrow and seriously limited notion about regrouping. He feels that regrouping is legal only when it is concerned with 10 or multiples of 10. He would be puzzled by the argument implied in the following:

$$\begin{array}{rrr} 14 & 7+7 & 15+1 \\ +\,18 \rightarrow & 9+9 \rightarrow & 15+1 \\ \hline & 16+16 & 30+2 = 32 \end{array}$$

because his thinking has been limited to:

$$\begin{array}{rrr} 14 & 10+4 & 20+0 \\ +\,18 \rightarrow & 10+8 \rightarrow & 10+2 \\ \hline & 20+12 & 30+2 = 32 \end{array}$$

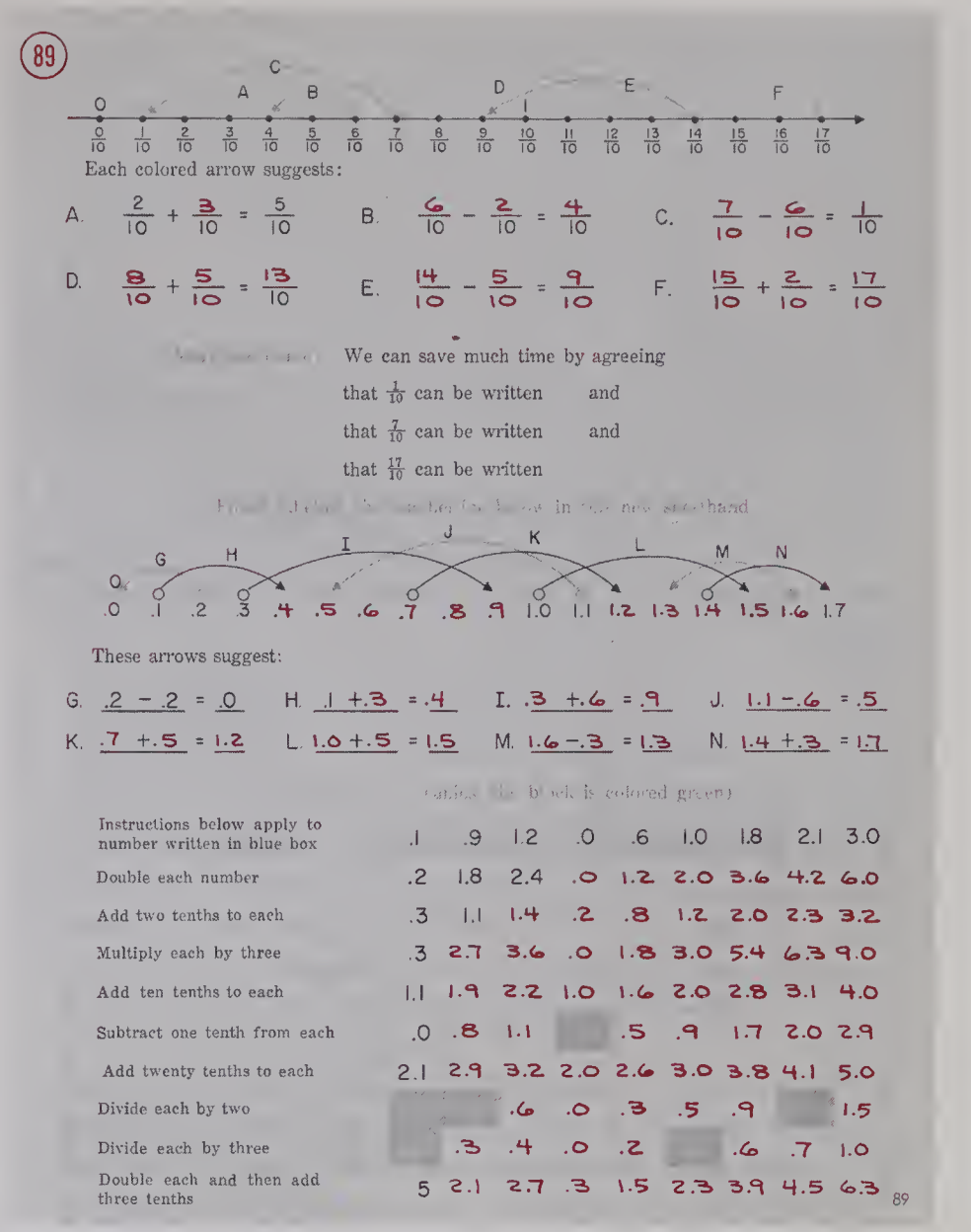
Certainly he will recognize some real economies in the second example of regrouping, and you can rest assured he will gladly choose it for its very economy.

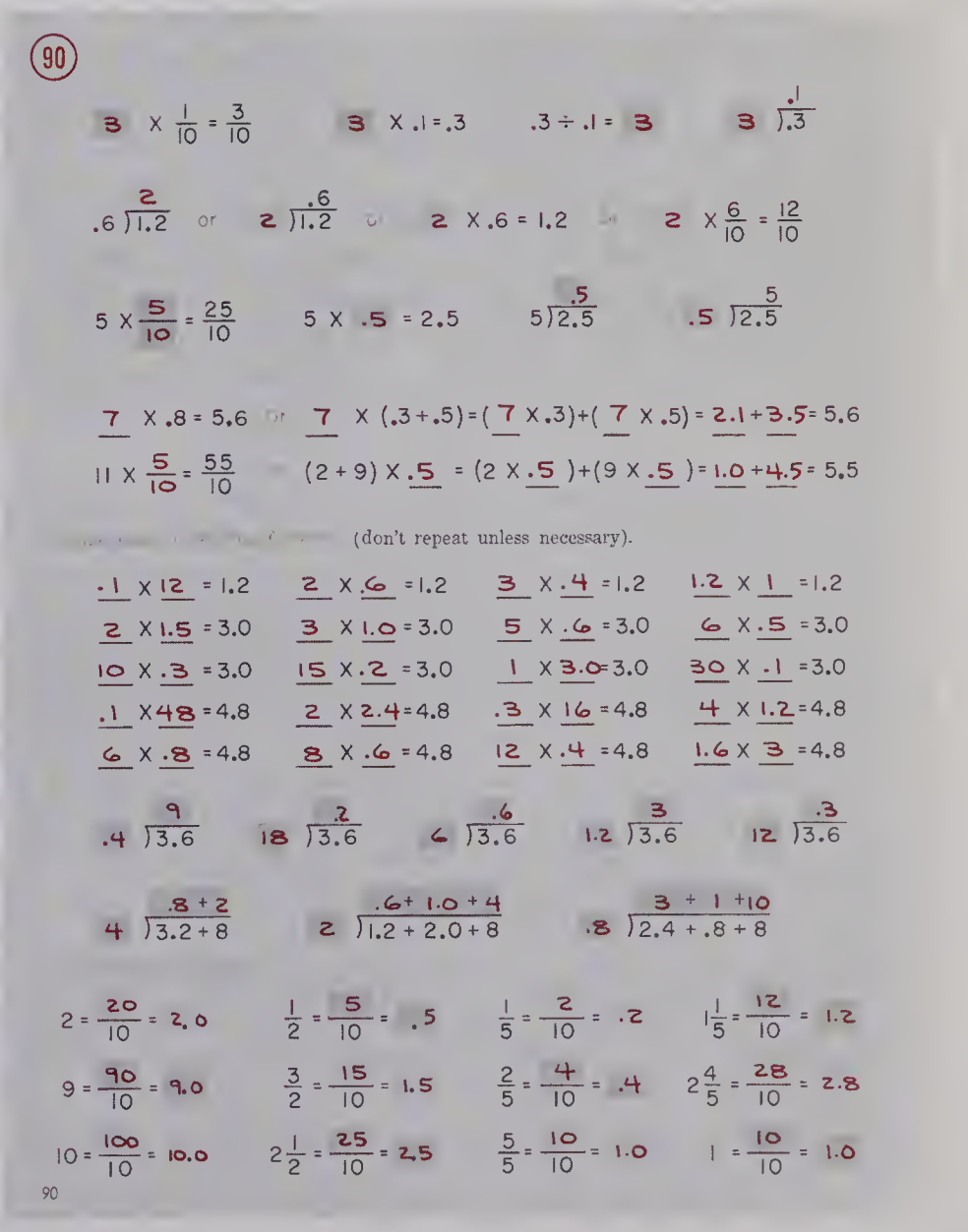
Each step along the way, the *Math Workshop* keeps the spotlight on understanding—on activities that lead to *discovery*—because deep and lasting understanding follows from discovery and not from memorization of rules or words.

Traditional methods

Children with a long background of emphasis on place value, with its carrying and borrowing of tens, hundreds and thousands, usually find the missing addend in 1007-996 in such manner as:

$$\begin{array}{r} {}^{\cancel{1}}{}^{9}{}^{1}{}^{\cancel{0}}{}^{\cancel{0}}7 \\ -\;\;9\;9\;6 \\ \hline 0\;1\;1 \end{array}$$


89
Each colored arrow suggests:
A. 2/10 + 3/10 = 5/10
B. 6/10 − 2/10 = 4/10
C. 7/10 − 6/10 = 1/10
D. 8/10 + 5/10 = 13/10
E. 14/10 − 5/10 = 9/10
F. 15/10 + 2/10 = 17/10
We can save much time by agreeing
that 1/10 can be written and
that 7/10 can be written and
that 17/10 can be written
These arrows suggest:
G. .2 − .2 = .0
H. .1 + .3 = .4
I. .3 + .6 = .9
J. 1.1 − .6 = .5
K. .7 + .5 = 1.2
L. 1.0 + .5 = 1.5
M. 1.6 − .3 = 1.3
N. 1.4 + .3 = 1.7
Instructions below apply to number written in blue box
.1 .9 1.2 .0 .6 1.0 1.8 2.1 3.0
Double each number
.2 1.8 2.4 .0 1.2 2.0 3.6 4.2 6.0
Add two tenths to each
.3 1.1 1.4 .2 .8 1.2 2.0 2.3 3.2
Multiply each by three
.3 2.7 3.6 .0 1.8 3.0 5.4 6.3 9.0
Add ten tenths to each
1.1 1.9 2.2 1.0 1.6 2.0 2.8 3.1 4.0
Subtract one tenth from each
.0 .8 1.1 .5 .9 1.7 2.0 2.9
Add twenty tenths to each
2.1 2.9 3.2 2.0 2.6 3.0 3.8 4.1 5.0
Divide each by two
.6 .0 .3 .5 .9 1.5
Divide each by three
.3 .4 .0 .2 .6 .7 1.0
Double each and then add three tenths
5 2.1 2.7 .3 1.5 2.3 3.9 4.5 6.3
89



Children who have been taught to look for the most convenient regrouping will argue:

$$1007 = 1000 + 7$$
$$996 = 1000 - 4$$

and see that the difference is 11 without further pause. And these children make fewer mistakes.

When a different kind or system of regrouping is required in 843-268, the argument turns on regrouping with the conveniences offered by the decimal system of notation:

$$\begin{array}{rrrr} 843 & 800+40+3 & 700+130+13 & \\ \underline{-268} & \underline{-200-60-8} & \underline{-200-\ 60-\ 8} & \\ & & 500+\ 70+\ 5 & = 575 \end{array}$$

This is an obvious use of the idea of place value. What, however, is the purpose of naming the idea and proceeding to talk about tens and hundreds? The usual purpose is in preparation for rules and prescribed procedures.

If the child does not have his own means to find the missing addend in 843-268, he is not ready to be concerned with place value. He needs more understanding of how to use the tools he already has. Once he learns a way of his own to find that missing addend—and it will involve regrouping—he can then compare his method with the one outlined above. If he is not impressed with its economy, he probably has an efficient method of his own.

Any generalizations that we want to make must be postponed until the subject matter to be generalized is already understood by the children. Algebra teachers know this so well! You cannot talk about algebra as the generalization of arithmetic if arithmetic is not already understood.

In the discovery method which has been proved so effective in learning mathematics, an important warning is that early verbalization about a discovery may "mutilate" the experience. Gertrude Hendrix (of the University of Illinois Committee on School Mathematics) chose the word "mutilate" as the most apt characterization of premature generalization. Professor Hendrix also found that a non-verbalized discovery is remembered and put to use as effectively without verbalization as it is with it.

In the case at hand, we introduce the usual notation for decimal fractions as a shorthand for a certain kind of common fraction and mixed numbers.

When there is indecision about such examples as $3 + 1.1 =$ ____, do not talk about rules. Rather, change the shorthand back to an expression written in common fractional notation:

$$3 + 1\ 1/10 = 3 + 11/10 = 30/10 + 11/10$$
$$= 41/10$$

or some such familiar argument. Our shorthand agreement tells us that 41/10 may be written as 4.1; so we come to the conclusion that

$$3 + 1.1 = 3.0 + 1.1 = 4.1$$

is a shorthand expression for the agreement above. If we were to write the above in vertical form, these is no question about "placement" of each addend with regard to the other:

$$\begin{array}{r} 3.0 \\ \underline{+\ 1.1} \\ 4.1 \end{array}$$

No other arrangement would make sense.

Our main concern, then, is to explore a shorthand in which to write sentences about fractions. As we proceed, we want to be sure that our results are reasonable—and can always check them by expressing them in familiar fractional notation. We do not expect mastery of this new notation at this time. It will be reintroduced again and again, and raised to a higher level of understanding each time.

Notes to accompany *Work Text* pages 91-93

A failure to understand the general idea of regrouping leads a pupil to feel comfortable when he sees the work on the left below, but a little uncomfortable with the work on the right:

$$\begin{array}{r} 34 \\ 8\overline{)272} \end{array} \qquad \begin{array}{r} 30 + \ 4 = 34 \\ 8\overline{)240 + 32} \end{array}$$

But a little scrutiny reveals that these are identical arguments. The squirming begins when the regrouping shows more freedom

$$\begin{array}{r} \\ 8\overline{)272} \end{array} = \begin{array}{r} 15 + 19 \\ 8\overline{)120+152} \end{array} = \begin{array}{r} 15 + 25 - \ 6 = 34 \\ 8\overline{)120+200-48} \end{array}$$

All the examples above depend on the idea that numbers have many names. Since we do not remember the other factor which, along with 8 as a factor, has 272 as a product, we find another name for 272 composed of more manageable terms. Then we find other factors which, combined with 8, have

these terms as a product. Finally, we add the factors—"to get the answer."

This is certainly an involved business either to be understood or taken on faith. When taken on faith, it leads to a very limited and rule-ridden ability to regroup. Consider:

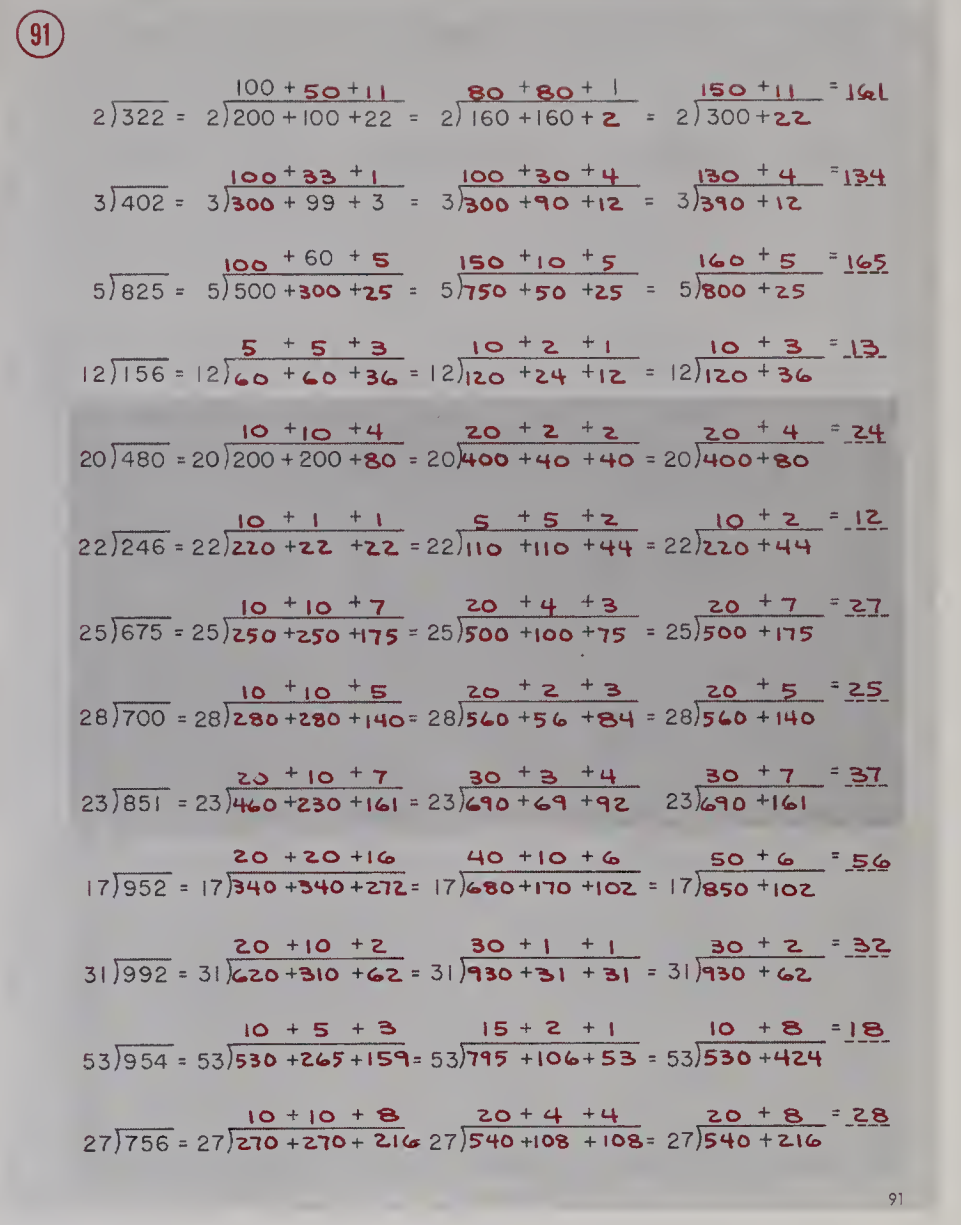
$7\overline{)693}$

If "regrouping" is understood as finding a more manageable name for 693, which of the following would be chosen?

$$\begin{array}{r} 90+\ 9 = 99 \\ 7\overline{)630+63} \end{array} \qquad \begin{array}{r} 100-1 = 99 \\ 7\overline{)700-7} \end{array}$$

The work on the left is dictated by the traditional algorism and, in many cases, is the efficient procedure. But it is not always the most efficient and, if it is the only kind of regrouping children ever see in division examples, it places narrow boundaries around one aspect of a truly basic idea of arithmetic, shutting out the broader expanses.

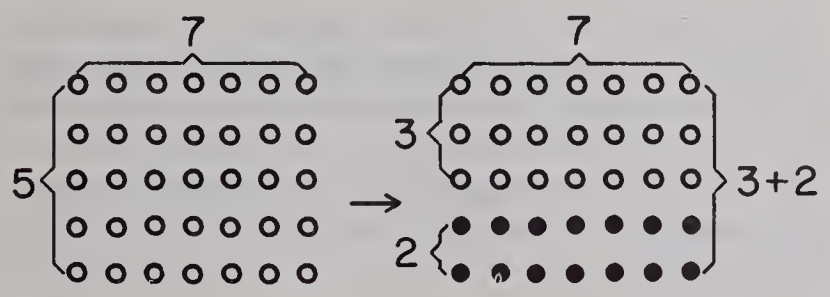
In anticipating the work on *Work Text* pages 91, 92 and 93, it might be well to recall Ben's "Do-It-to-Both" rules that mathematicians refer to as the "distributive principle."



$7\times5 = 7(2+3) = (7\times2) + (7\times3) = 14+21 = 35$
$7\times5 = (3+4)5 = (3\times5) + (4\times5) = 15+20 = 35$
$7\times5 = 7(1+4) = (9\times1) + (7\times4) = 7+28 = 35$

$7\times5 = (5+2)5 = (5\times5) + (5\times2) = 25+10 = 35$

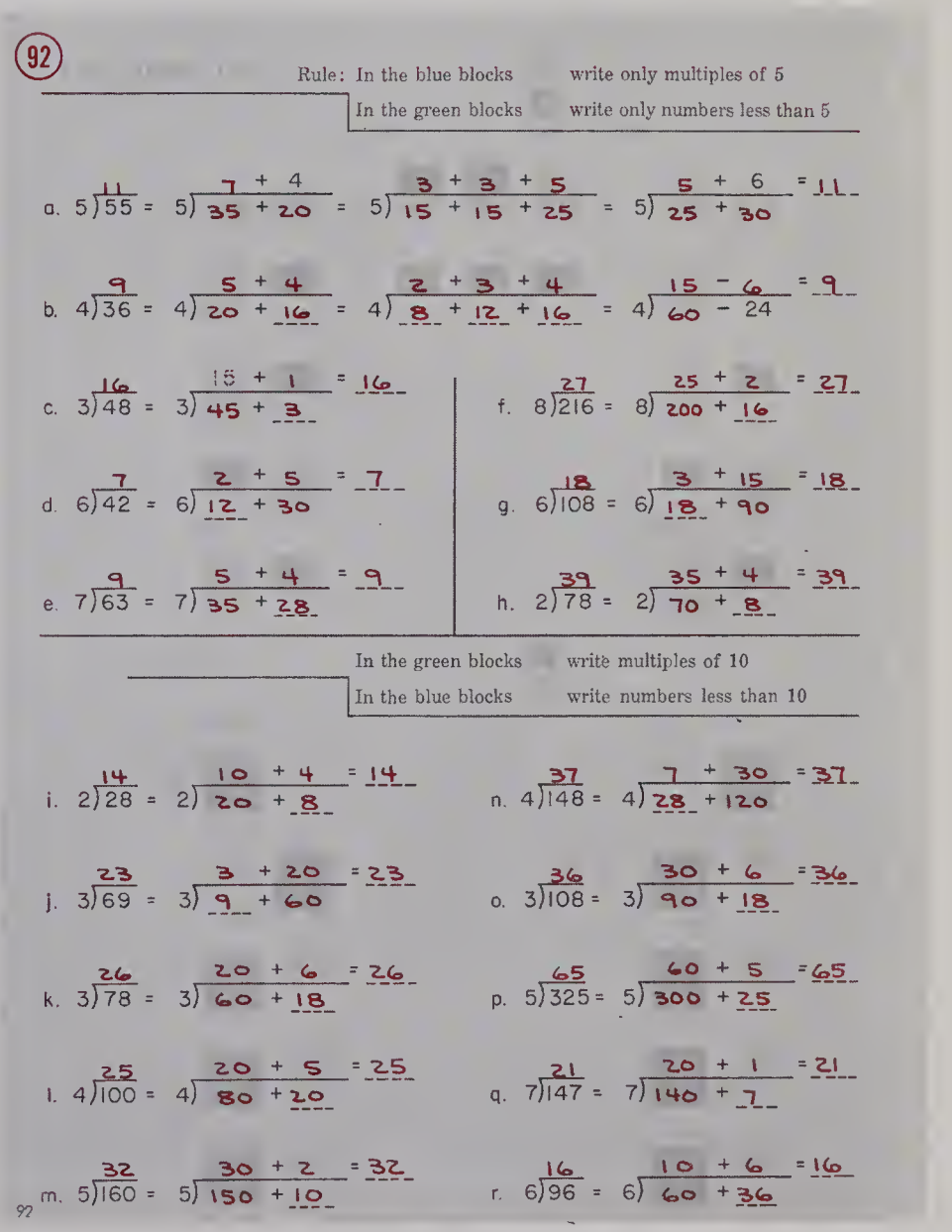
Draw a sketch to illustrate some of these examples:

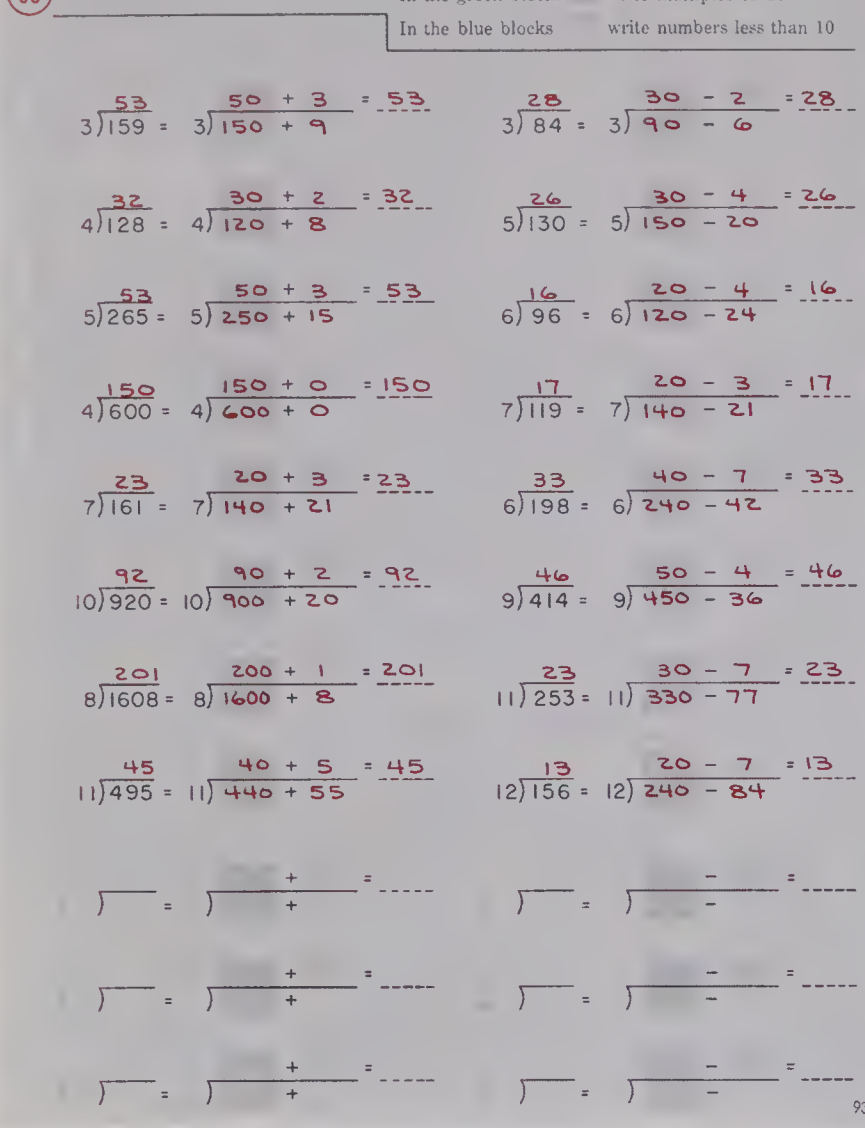


Seven columns of 5 counters is the same as 7 columns of 3 counters plus 7 columns of 2 counters, or 21 counters plus 14 counters.

$$7\times5 = 7\times(3+2) = 7\times3 + 7\times2 = 21+14 = 35$$

Consider the reverse argument for this sketch. The 35 counters are, on the left, "regrouped" into 21 and 14 counters, each group arranged in 7 columns. If 21 are arranged in 7 columns, there must be 3 rows in this group. If 14 are arranged in





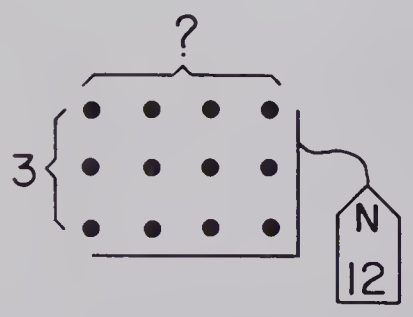
7 columns, there must be 2 rows in this group. So, there are 7 columns in all and 3 plus 2 rows, or 5 rows. Thus, 35 counters arranged in 7 columns must have 5 in each column—or, there must be 5 rows.

$$35 \div 7 = (21+14)7 = 21 \div 7 + 14 \div 7 = 3+2 = 5$$

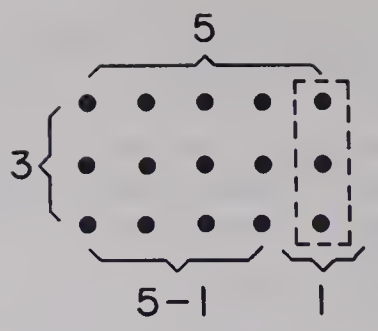
This argument might be written in another form:

$$7\overline{)35} = \quad \overset{3+2=5}{7\overline{)21+4}}$$

A variation on this same idea might be suggested by the following example: Find the missing factor indicated by the expression, $12 \div 3$. A sketch will help us see the idea:



Since we know that 15 counters can be arranged in three rows with 5 in a row, we draw this sketch:



Now, we know that $12 = 15 - 3$, so we remove one column of 3 counters from our consideration and have $5 - 1$ columns left. So, $12 \div 3 = 5 - 1 = 4$. We can summarize the reasoning as follows:

$$12 \div 3 = (15-3) \div 3 = (15 \div 3) - (3 \div 3) = 5 - 1 = 4$$

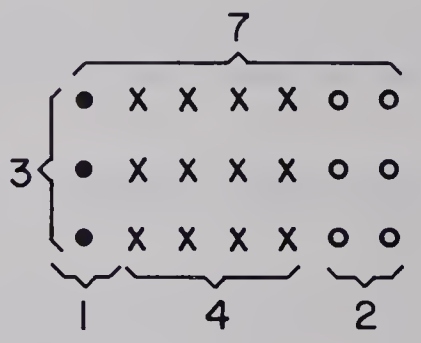
And this is the pattern in an example above—which proved to be very practical:

$$693 \div 7 = (700-7) \div 7 = 700 \div 7 - (7 \div 7)$$
$$= 100 - 1 = 99$$

Extend this development by beginning with an example of Ben's "Do-It-to-Both" rule:

$$3 \times 7 = 3(1+4+2) = 3 \times 1 + 3 \times 4 + 3 \times 2$$
$$= 3 + 12 + 6 = 21$$

A sketch:



Now, turn the argument around. We have 21 counters in 3 rows. We "regroup" our counters into groups of 3 and 12 and 6, as indicated in the sketch. We find that the 3 counters can be put in 1 column; the 12 can be arranged in 4 columns, and the remaining 6 in 2 columns. Finally, $1 + 4 + 2 = 7$. Summarizing:

$$21 \div 3 = (3+12+6) \div 3 = 3 \div 3 + 6 \div 3$$
$$= 1 + 4 + 2 = 7$$

This last example seems quite obvious and a long way 'round at that. But, before dismissing it, consider a more challenging example:

$$\overset{100+\ 40+3}{3\overline{)429} = 3\overline{)300+120+9}}$$

This example points up the fact that the steps

in the traditional algorism of division and pattern of procedures is exactly the same as the attack made above on 21 ÷ 3 = ____. So, while the arithmetic of the latter is trivial and obvious, the procedure is, step by step, the precise procedure with which we are here vitally concerned. The only difference lies in the fact that, in finding the missing factor indicated by 429 ÷ 3, we selected a particular system for regrouping—a special scheme to guide us in our search for a more manageable name for 429.

The *Work Text* pages

Page 91 is devoted exclusively to regrouping in many different ways. If we insist that each example be attacked in three different ways, we are sure that everyone, including those who have already been taught a particular scheme of regrouping, will see some of the freedom we have in selecting other names for a number.

After the page is completed, ask for volunteers to explain what they think is the most efficient method of regrouping in each of the 13 examples.

At the top of page 92, a division game is outlined. Certain rules are given to be followed in finding other names for numbers. In the blue areas, only names for multiples of 5 are permitted. In the green areas, only names for numbers less than 5. These rules require foresight in regrouping and, in some cases, some ingenuity (third example in line *b*).

The division game at the top of the page anticipates the other game on the page. There is a simple rule change in it—we are now limited to multiples of 10 and to numbers less than 10.

The result of this change in rules is to require that every example be completed in a way that is an exact parallel of the traditional algorism.

Page 92 continues this game, working up to examples H, O and P in which the given factor is a "2-place" numeral.

It would be difficult for a child to get this far without having deepened his understanding of regrouping for division. He would also begin to see the convenience of following the rules in the last game most of the time.

At the bottom of page 93, each child can make up his own examples. With a prompt from the teacher about "showing what you can do," most children will select examples that will indicate the extent of their understanding of the procedures of division.

Notes to accompany *Work Text* page 94

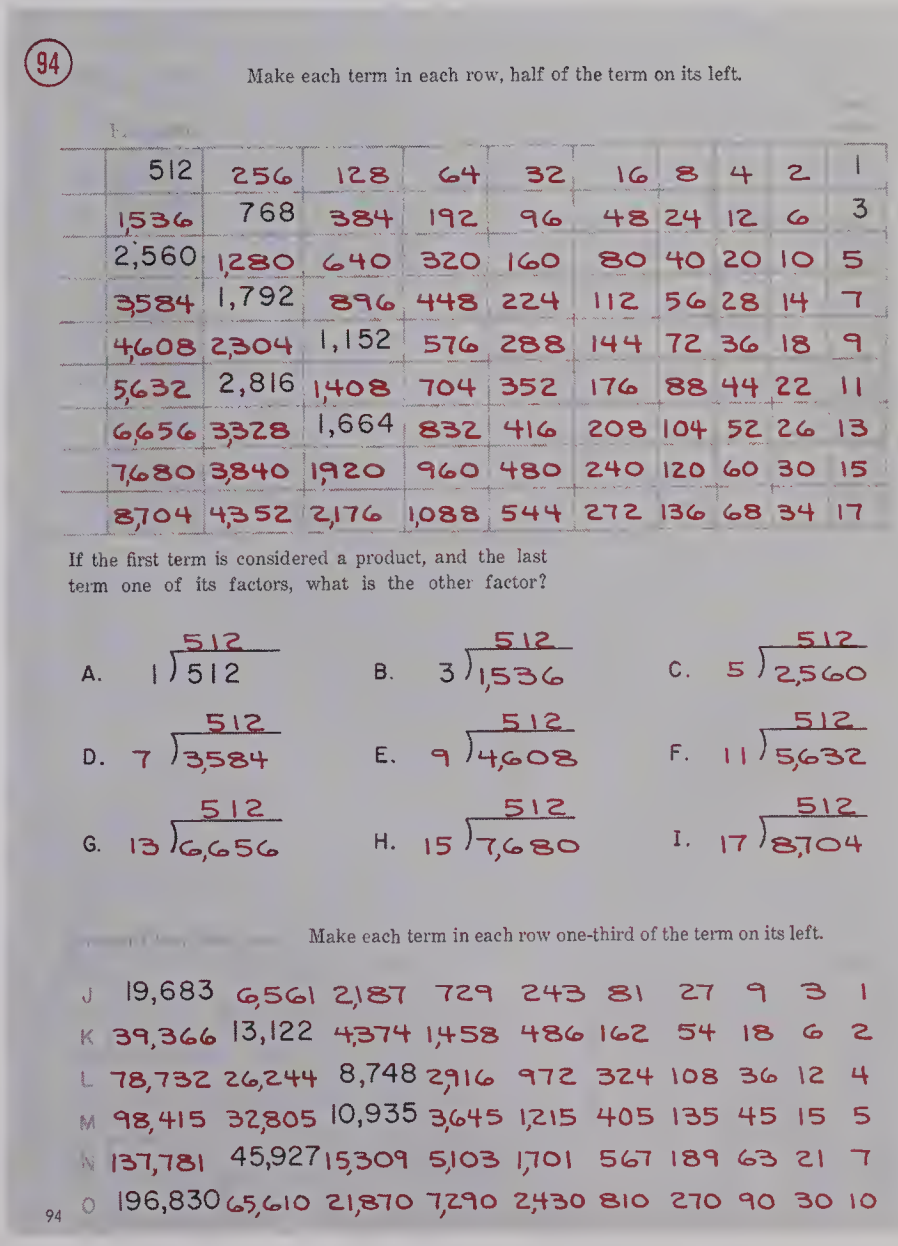
"Chain reactions" such as those on page 94, are far more than disguised drill (though that is certainly one of their purposes).

With a starting point and a simple rule, the "reaction" gets underway. Each additional step depends upon the preceding step, a fact that may provide some dramatic lessons concerning the effect of a "little" or a "careless" mistake. Unless every step is flawless, the whole reaction is disrupted.

This is pointed up sharply in examples A and B on this page. Each term is half its predecessor. If any mistake is made along the way, no matter how slight, the next-to-last terms in these examples will not be 2 and 6, respectively.

Further, patterns can be built into a series of chain reactions, such as A through G. The last terms generate the sequence of odd numbers. Many children will notice this. Some will assume it holds, and work backwards, from right to left.

When they see the pattern, this becomes the key for examples H and I. Their last terms are 15 and 17 . . . "and now you can reconstruct the chain reactions."



94

Make each term in each row, half of the term on its left.

	512	256	128	64	32	16	8	4	2	1
	1,536	768	384	192	96	48	24	12	6	3
	2,560	1,280	640	320	160	80	40	20	10	5
	3,584	1,792	896	448	224	112	56	28	14	7
	4,608	2,304	1,152	576	288	144	72	36	18	9
	5,632	2,816	1,408	704	352	176	88	44	22	11
	6,656	3,328	1,664	832	416	208	104	52	26	13
	7,680	3,840	1,920	960	480	240	120	60	30	15
	8,704	4,352	2,176	1,088	544	272	136	68	34	17

If the first term is considered a product, and the last term one of its factors, what is the other factor?

A. $1\overline{)512}$ 512

B. $3\overline{)1,536}$ 512

C. $5\overline{)2,560}$ 512

D. $7\overline{)3,584}$ 512

E. $9\overline{)4,608}$ 512

F. $11\overline{)5,632}$ 512

G. $13\overline{)6,656}$ 512

H. $15\overline{)7,680}$ 512

I. $17\overline{)8,704}$ 512

Make each term in each row one-third of the term on its left.

J	19,683	6,561	2,187	729	243	81	27	9	3	1
K	39,366	13,122	4,374	1,458	486	162	54	18	6	2
L	78,732	26,244	8,748	2,916	972	324	108	36	12	4
M	98,415	32,805	10,935	3,645	1,215	405	135	45	15	5
N	137,781	45,927	15,309	5,103	1,701	567	189	63	21	7
O	196,830	65,610	21,870	7,290	2,430	810	270	90	30	10

Then a nice surprise to the question given below the chart—the same answer in every example.

Another interesting fact is that most chain reactions are open on both ends for the more able children to follow on their own. For instance, example A can be extended to the left in this way:

2048, 1024, 512, __ __ (256) __ __

And it can also be extended to the right:

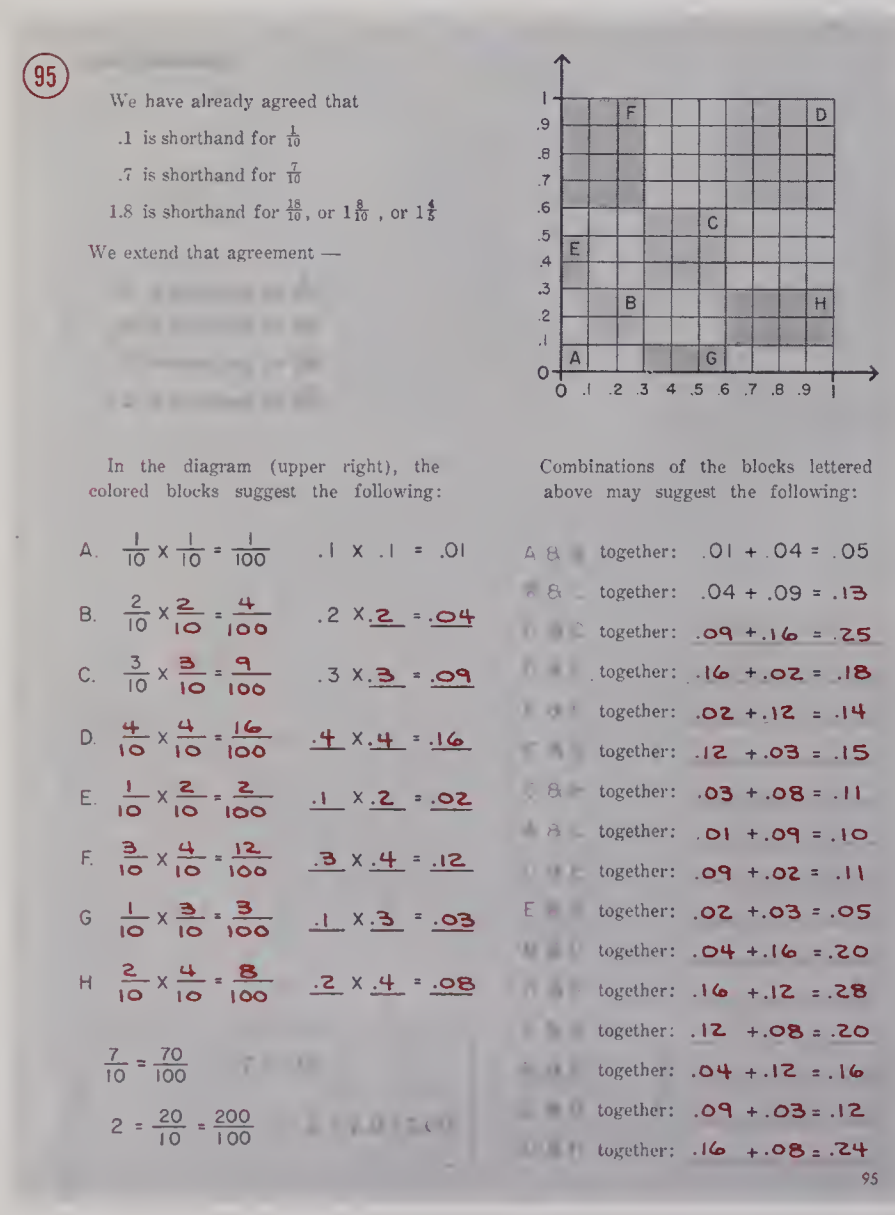
2 , 1, 1/2, 1/4, 1/8, 1/16 →

The same holds for all the examples.

Thus, though chain reactions require many computations, they *move,* they go somewhere, patterns emerge, there are surprises—they are open-ended.

Notes to accompany *Work Text* page 95

We return on page 95 to expand the *shorthand* of decimal fractions. The examples on pages 89 and 90 were concerned with a short method for writing fractions with 10 as the denominator; here, we consider fractions with 10 or 100 as denominators.


95
We have already agreed that
.1 is shorthand for 1/10
.7 is shorthand for 7/10
1.8 is shorthand for 18/10, or 1 8/10, or 1 4/5
We extend that agreement —
In the diagram (upper right), the colored blocks suggest the following:
A. 1/10 x 1/10 = 1/100 .1 x .1 = .01
B. 2/10 x 2/10 = 4/100 .2 x .2 = .04
C. 3/10 x 3/10 = 9/100 .3 x .3 = .09
D. 4/10 x 4/10 = 16/100 .4 x .4 = .16
E. 1/10 x 2/10 = 2/100 .1 x .2 = .02
F. 3/10 x 4/10 = 12/100 .3 x .4 = .12
G 1/10 x 3/10 = 3/100 .1 x .3 = .03
H 2/10 x 4/10 = 8/100 .2 x .4 = .08
7/10 = 70/100
2 = 20/10 = 200/100
A
B
C
D
E
F
G
H
Combinations of the blocks lettered above may suggest the following:
A & B together: .01 + .04 = .05
B & C together: .04 + .09 = .13
C & D together: .09 + .16 = .25
D & E together: .16 + .02 = .18
E & F together: .02 + .12 = .14
F & G together: .12 + .03 = .15
G & H together: .03 + .08 = .11
A & D together: .01 + .09 = .10
C & E together: .09 + .02 = .11
E & B together: .02 + .03 = .05
H & C together: .04 + .16 = .20
D & F together: .16 + .12 = .28
F & H together: .12 + .08 = .20
H & B together: .04 + .12 = .16
C & G together: .09 + .03 = .12
D & H together: .16 + .08 = .24
95

The sketch at the top of the page is a suggestion of the idea of 1×1, and we know that $1 \times 1 = 1$. Each of the two number lines has 10 divisions between 0 and 1. They are labeled in the shorthand. (Refer to the number line at the top of *Work Text* page 89 in which common fractions were used as labels.)

If the sketch suggests that $1 \times 1 = 1$, the blue area labeled A suggests the idea of $.1 \times .1$, and we can see that it is 1/100 of the entire sketch because there are 100 such areas in the whole sketch.

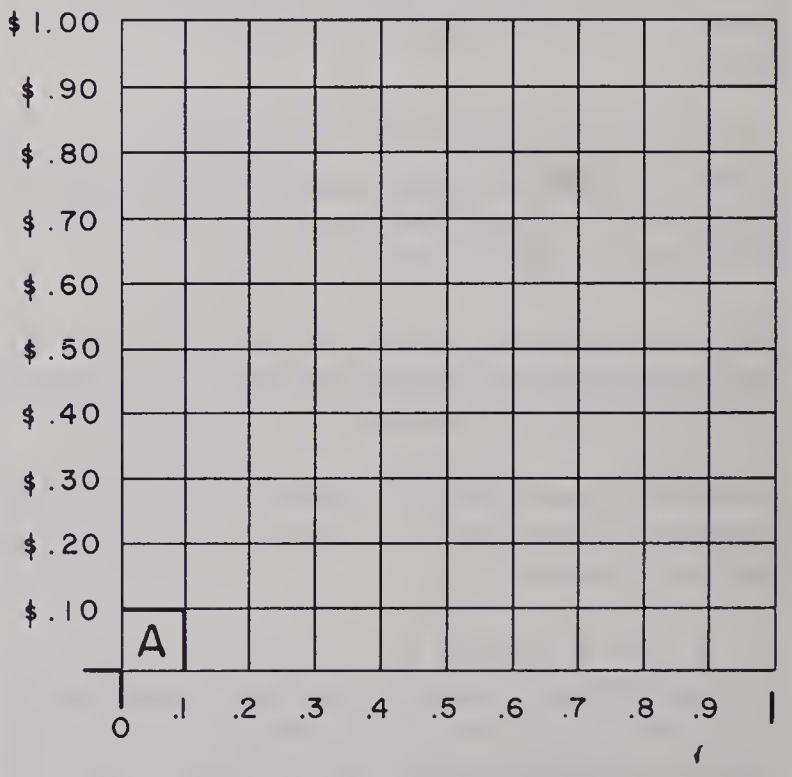
Our shorthand is expanded "by agreement," so we can write $1/100 = .01$ and we see that section A indicates that:

$$.1 \times .1 = .01$$

Each of the labeled parts of the diagram should be considered individually and described in the series of statements on the left of the page.

On the right, combinations of the areas is considered. The first examples ask: If A suggests .01 and B suggests .04, together they suggest that .01 + .04 = .05 . . . etc.

When this page is completed, draw a similar grid on the board with a slight change in one of the scales:



Now the whole area suggests 1 times $1.00, or $1.00. If that is agreed upon, then the area labeled A suggests the idea of .1 x $.10 or "one-tenth of ten cents." The area is clearly 1/100 of the whole (dollar, in this case); it suggests the idea of 1¢

"It is true that 1¢ is one-tenth of a dime?"

And, with similar analysis, you can find that "two-tenths of half a dollar is one-tenth of a dollar:

$$.2 \times .5 = .10$$

and "half of a half dollar is a quarter, or 25¢:

$$.5 \times .5 = .25$$

As each of these expressions is developed, use it as a headline:

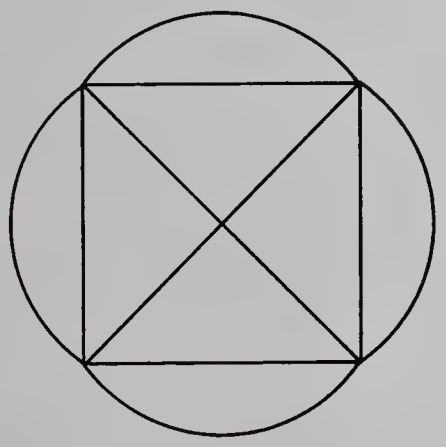
$$.5 \times .5 = .25$$

"Bill and Jim decided to divide the half dollar profit in the sale equally, so each got a half a half dollar . . . each got a quarter, or 25¢."

"Mary lived a half mile from school. Half-way to school she stopped to visit her aunt. She had walked a quarter of a mile and had a quarter of a mile left to go."

Notes to accompany *Work Text* page 96

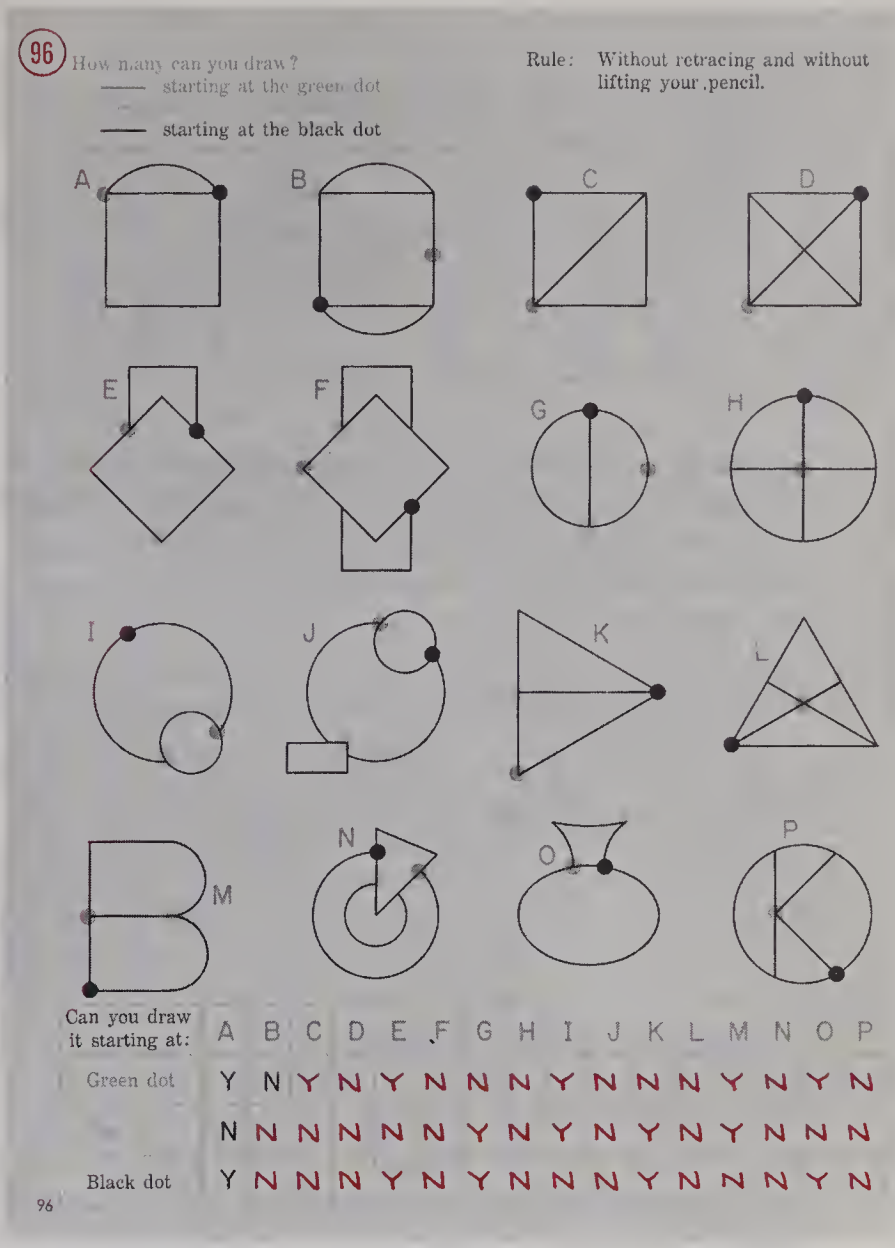
Here is a change of pace that is reminiscent of an old puzzle that is an expanded combination of examples B and D on the page. The puzzle challenged you to draw the following diagram without lifting your pencil and without retracing any part of a line.



Many generations have struggled hours to meet this challenge. After much trial, and just as much error, each experimenter reluctantly guesses that it is probably impossible.

We are not going to prove the impossibility of that particular problem at this point. However, it can be proved to be impossible by mathematics. We are exploring here easier problems.

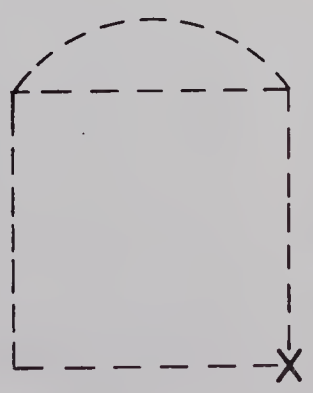
Children will be able to see in example A that the drawing can be done according to the rules if you start at the green dot or the black dot. But, can it be done if you start at the blue dot? Answer: No. Question: Why not?


96
How many can you draw?
starting at the green dot
starting at the black dot
Rule: Without retracing and without lifting your pencil.
A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P
Can you draw it starting at:
A B C D E F G H I J K L M N O P
Green dot
Y N Y N Y N N N Y N N N Y N Y N
N N N N N N Y N Y N Y N Y N N N
Black dot
Y N N N Y N Y N N N Y N N N Y N
96

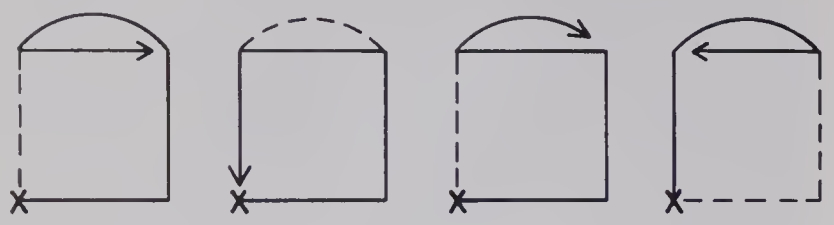
At the bottom of the page, a record is to be kept. When the diagram being considered can be drawn according to the rules by starting at the dot indicated, indicate the fact with a "Y" in the appropriate cell. If no way is found, do not record this fact . . . someone else may find a way, and you would have to change the record.

Assure children that we are not concerned at all with a "trick." We are testing our ability to see ahead.

To test a pupil's assertion that he can solve a particular example, sketch the outline of the shape on the board with a light broken line and mark the starting point with a cross. Suppose Bill says he can draw example A according to the rules and beginning at the blue dot. A test sketch would be:

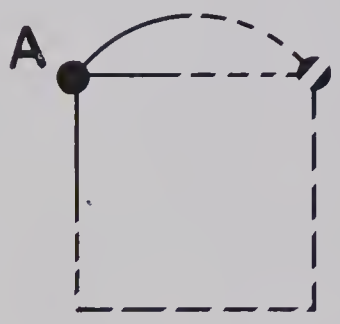


He will run into trouble, no matter which of the following courses he takes:



The arrow indicates the point at which his efforts will prove to have failed. But, if he starts at either the green or black dot, he can succeed. You may want to leave this to be explored by a few. Or, you may want to give hints of an answer. Such as—

Consider the corner marked by the green dot in A:



There are three lines leading out from, or into, this point. If you start at the dot, you will go out along one line and come back along another—and you will leave by way of the third line . . . never to return. So if you *start* at such a corner, you can never *end* there.

But suppose you start somewhere else in the diagram. Now you will come to the point on one of the lines, leave it by a second line, and return to it by the third line—and you can not get away again . . . you have used up all three lines. So, if you *start* away from such a point, you must *end* at that point.

Thus, if there are two such points in example A, you must start at one (away from the other)and end at the other (away from the starting point).

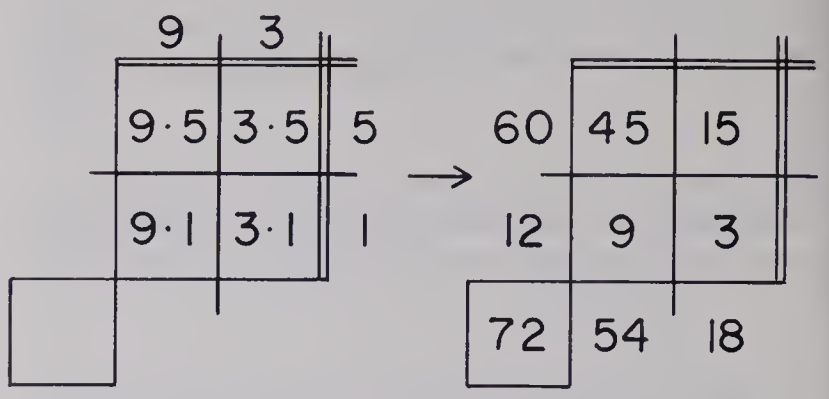
But consider your problem if you start at the blue dot. You are starting away from both points where three lines meet, so you must end *at both*! . . . which is clearly impossible.

Notes to accompany *Work Text* page 97

Page 97 provides another study of regrouping—this time in multiplication. Each example may be considered as a record of Ben's "Double-Do-It-to-Both" rule that leads him to the following argument:

$$12 \cdot 6 = (9+3)(5+1) = 9\cdot5 + 9\cdot1 + 3\cdot5 + 3\cdot1$$
$$= 45 + 9 + 15 + 3 = 72$$

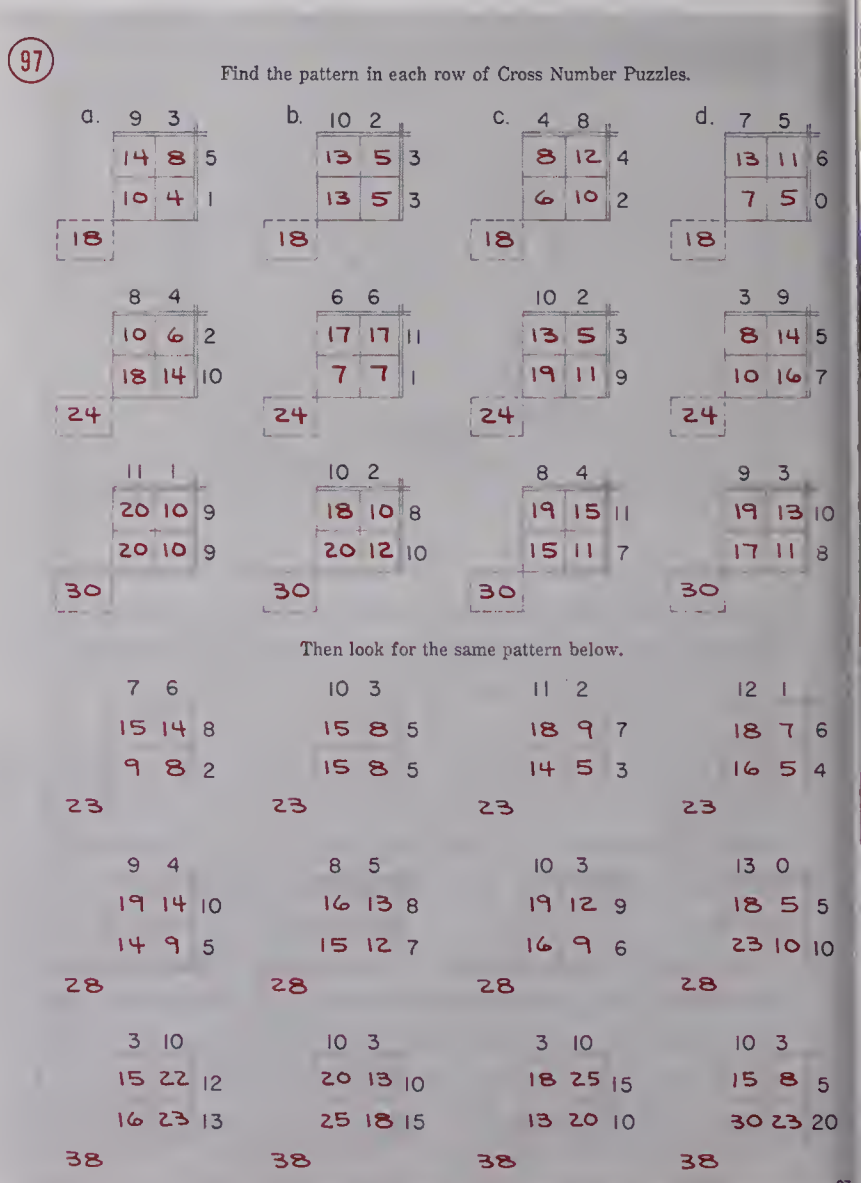
and the first Cross Number Puzzle in column *a* keeps a record of these computations:



When the top row of four puzzles is completed, discuss the development that 72 is repeated in each check box. Why? Because each is simply another way of regrouping to find the product of 12 and 6. They can be written as follows:

$12 \times 6 = (9+3)(5+1) = 45 + 9 + 15 + 3 = 72$
$12 \times 6 = (10+2)(3+3) = 30 + 30 + 6 + 6 = 72$
$12 \times 6 = (4+8)(4+2) = 16 + 8 + 32 + 16 = 72$
$12 \times 6 = (7+5)(6+0) = 42 + 0 + 30 + 0 = 72$

The second row turns out to be a similar study in regrouping to find the product of 12 and 12, or 144.



$12 \times 12 = (8+4)\,(2+10)$
$12 \times 12 = (6+6)\,(11+1)$
$12 \times 12 = (10+2)\,(3+9)$
$12 \times 12 = (3+9)\,(5+7)$

And the third row explores several ways to regroup each factor in the expression: $12 \times 18 = 216$.

Before proceeding, ask if anyone sees another pattern in this series of 12 Cross Number Puzzles.

First row studies regrouping factors of 72
Second row studies regrouping factors of 144
Third row studies regrouping factors of 216
(Secret: 72 + 144 = 216)

Each column of 3 examples might be summarized in this way:

$(12\times18) = (12\times6) + (12\times12) = 72 + 144 = 216$

which is, of course, another example of the same pattern that has been under study.

The 12 puzzles in the lower section of page 97 follow an almost identical overall pattern. After they have been completed, each column of 3 examples can be summarized as:

$(13\times25) = (13\times10) + (13\times15) =$
$130 + 195 = 325$

Only two examples on this page regroup in accordance with the procedures of the traditional algorism —the third from the top in column *b* and the bottom example in column *d*.

Direct the discussion to this fact just noted. "Of the many ways we can regroup, why do we usually use the pattern of these two particular examples?" Discuss the following as very similar procedures to the Cross Number Puzzles under consideration:

12	18	13	25
× 18	× 12	× 25	× 13
96	36	65	75
12	18	26	25
216	216	325	325

Explore the Cross Number Puzzle approach to an example in which one factor is written as a 3-place numeral: 253 × 15

	200	50	3	
1,265	1,000	250	15	5
2,530	2,000	500	30	10
3,795	3,000	750	45	

	125	92	36	
1,518	750	552	216	6
2,277	1,125	828	324	9
3,795	1,875	1,380	540	

Such investigation can be extended, indicating the usually more efficient regrouping, but always showing that there are an unlimited number of other regroupings possible.

Notes to accompany *Work Text* page 98

Each example on page 98 is concerned with a particular way of indicating a reduction in price. We might call this one of the "sale" or "price reduction" devices that apply at once to many items. "Reduced from $7.98 to $6.25", however, is a reference to a single item. It is different.

Are there other methods of reducing the price of a large number of items—at one time—other than the one used on this page?

"All items on this counter only 50¢."
"One dollar off every item on this counter."

98

In preparing for a sale, the clerk in the store had to compute many new prices. Here are some of the records he made:

Regular price	$ 4.28	$ 3.00	$ 3.50	$ 10.82	$124.30
Amount of reduction	$2.14	$1.50	$1.75	$5.41	$62.15
Sale price	$2.14	$1.50	$1.75	$5.41	$62.15

Regular price	$ 3.60	$ 4.50	$ 6.81	$ 123.75	$ 51.87
Amount of reduction	$1.20	$1.50	$2.27	$41.25	$17.29
Sale price	$2.40	$3.00	$4.54	$82.50	$34.58

Regular price	$ 8.60	$ 2.68	$ 9.28	$ 45.76	$ 150.00
Amount of reduction	$2.15	$ .67	$2.32	$11.44	$37.50
Sale price	$6.45	$2.01	$6.96	$34.32	$112.50

Regular price	$ 1.90	$ 35.00	$ 10.00	$ 100.00	$ 848.00
Amount of reduction	$ .19	$3.50	$1.00	$10.00	$84.81
Sale price	$1.71	$31.50	$9.00	$90.00	$763.29

Regular price	$ 7.50	$ .96	$ 12.87	$ 225.00	$ 29.34
Amount of reduction	$5.00	$ .64	$8.58	$150.00	$19.56
Sale price	$2.50	$ .32	$4.29	$75.00	$ 9.78

"Two for 1¢ more than the regular price of one."

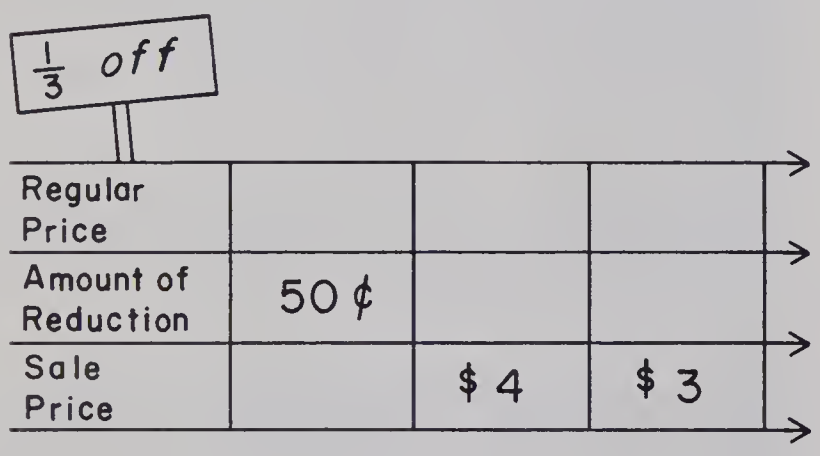
"Trade in your old vacuum cleaner for $30 credit." (It is assumed that the trade-in is worth much less than $30, so the offer is, in effect, a reduction on the price of any cleaner that is being sold.)

"What are other ways of indicating the kind of reduction in price considered on page 98?"

"You pay only 1/2 the price."
"Two for the price of one."
"50% off."
"Pay only 50%."
"Half price."
"Save one-half."
. . .all simply variations of the first example on page 98.

The class might cut out of newspapers some examples of expressions used to advertise price reductions. This will usually provoke a discussion about the meaning of 50%. Explain that this is simply another way of writing 50/100 . . . and it seems quite clear that the per cent sign is a rearrangement of the parts of the numeral 100 . . . 1,0,0 → 0/0 → %. Further, "per cent" is derived from "per hundred" (century). Since 50/100 = 1/2, then 50% and 1/2 are simply two names for the same number. A similar discussion of 10%, 25%, 15%, etc., is in order.

An extension of the example on page 98 can be developed from similar charts, but with different information provided:



Each of the examples can be extended in this way—some with signs indicating the reduction as "25% off," "3 for the price of 2," "$2.50 off each item," "Pay only 60%," etc.

Notes to accompany *Work Text* page 99

Have the children begin this page with only the barest of introduction. As soon as everyone recognizes the problem, let each move ahead at his own speed—in the area not shaded green.

When some are nearing completion of this unshaded area, initiate a group discussion.

"Have you found any shortcuts that you can rely on?" Some are obvious.

"Any number that ends in 0 is divisible by 10—or has 10 as a factor."

"And any number that ends in 5 or 0 has 5 as a factor."

"All even numbers are divisible by 2—have 2 as a factor."

Others are less obvious. List all the numbers involved that are divisible by 3—and note the sum of the digits:

÷ by 3	24	60	210	120	720	72	2310	729
Sum of digits	6	6	3	3	9	9	6	18

"What do you notice about the sum of the digits in each case?" Answer: They are multiples of 3.

"Is it true that if a number is divisible by 3, the sum of its digits is divisible by 3? If that is true, then 1,000,101 must be divisible by 3."

The search is on—for an example in which the rule fails . . . but one cannot be found.

"Since 9 = 3 × 3, perhaps a similar pattern will be revealed in a study of the multiples of 9."

Multiples of 9	720	72	729	216
Sum of digits	9	9	18	9

"Will the sum of the digits of all multiples of 9 be themselves multiples of 9?" . . . and another search is on. The generalization is true. When the sum of the digits turn out to be 18 or 27 or 36, 252, etc., we know the sum is also divisible by 9 simply by adding the digits: 18 → 9, 27 → 9, 36 → 9, 252 → 9, etc. (This, of course, is the principle that leads to the procedure called "casting out 9's.")

"Is there a shortcut to finding whether a number is divisible by 4?"

Of course, it will have to be an even number, but 210 is an even number that is not divisible by 4. Six and 10 and 14 are other examples.

"Since all numbers less than 100 are easily tested by inspection, let's consider all numbers in the list that are greater than 100:

	210	120	720	2310	256	729	216	4290
Yes or No	N	Y	Y	N	Y	N	Y	N

"Is there any simple pattern? Write them in two groups and underline the last 2 digits on the right—tens and units columns."

Yes—1<u>20</u>, 7<u>20</u>, 2<u>56</u>, 2<u>16</u>
No —2<u>10</u>, 23<u>10</u>, 7<u>29</u>, 42<u>90</u>

"No more limits."

If a number is divisible by four, the number named by its last two digits on the right is divisible by 4. This is a result of the fact that any number can be written as so many hundreds plus a number less than 100:

4,290 = 4,200 + 90
362, 880 = 362,800 + 80

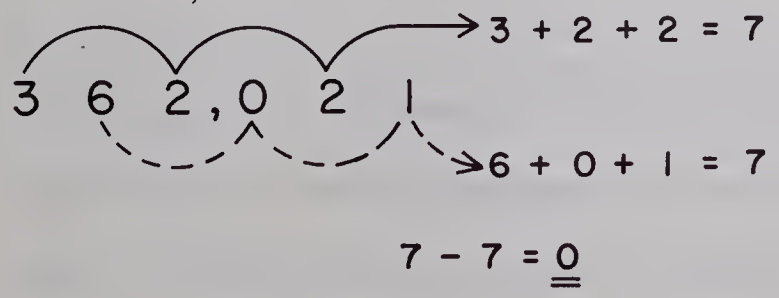
Since 4 is a factor of 100, any number ending in two zeros is divisible by 4, and we need only inspect these last two digits to test whether or not the number is divisible by 4.

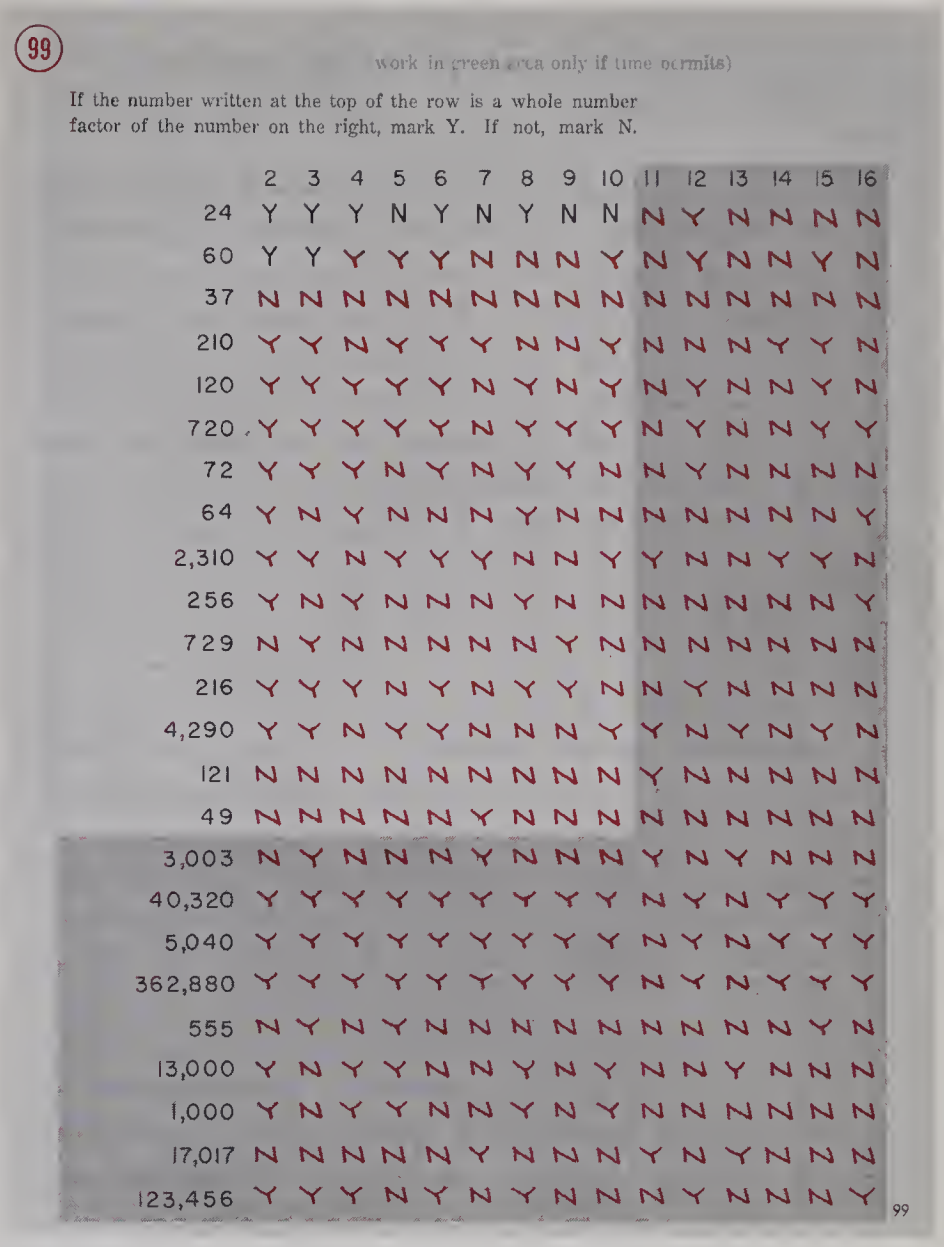
"Any shortcuts for divisibility by 6?"
Here, we take a new and most important tack in finding a shortcut. Every number that is divisible by 6 must have 6 as a factor—and 6 = 2·3—so every number divisible by 6 has 2 as a factor and 3 as a factor. We already know shortcuts for both—so, every even number is divisible by 6 if the sum of its digits is divisible by 3.

The tests for divisibility by 7 are so complicated as to be useless in this context. However, this problem is fascinating at a higher level of sophistication.

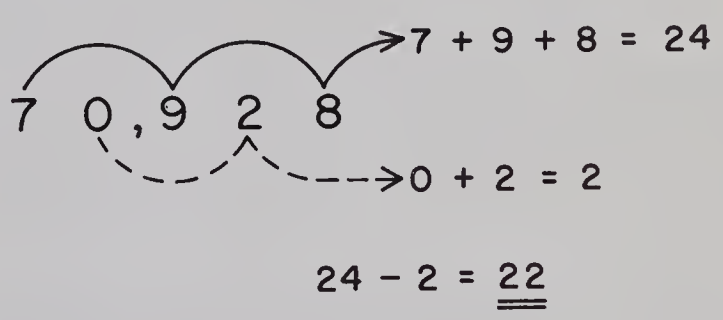
The test for divisibility by 8 is an expansion of the idea of the test for 4. Since 1,000 is divisible by 8, all multiples of 1,000 are divisible by 8. We can write the number 329,136 as 329,000+136. Since we know that 329,000 is divisible by 8, we need only inspect 136 for divisibility. Since 8 is a factor of 136, it is also a factor of any number ending in 136 such as 329,136, or 11,136.

The test for 11 is unusual. Beginning at either end of a numeral, find the sum of every other digit. Consider 362,021:




99
work in green area only if time permits)
If the number written at the top of the row is a whole number factor of the number on the right, mark Y. If not, mark N.
2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
24 Y Y Y N Y N Y N N N Y N N N N
60 Y Y Y Y Y N N N Y N Y N N Y N
37 N N N N N N N N N N N N N N N
210 Y Y N Y Y Y N N Y N N N Y Y N
120 Y Y Y Y Y N Y N Y N Y N N Y N
720 Y Y Y Y Y N Y Y Y N Y N N Y Y
72 Y Y Y N Y N Y Y N N Y N N N N
64 Y N Y N N N Y N N N N N N N Y
2,310 Y Y N Y Y Y N N Y Y N N Y Y N
256 Y N Y N N N Y N N N N N N N Y
729 N Y N N N N N Y N N N N N N N
216 Y Y Y N Y N Y Y N N Y N N N N
4,290 Y Y N Y Y N N N Y Y N Y N Y N
121 N N N N N N N N N Y N N N N N
49 N N N N N Y N N N N N N N N N
3,003 N Y N N N Y N N N Y N Y N N N
40,320 Y Y Y Y Y Y Y Y Y N Y N Y Y Y
5,040 Y Y Y Y Y Y Y Y Y N Y N Y Y Y
362,880 Y Y Y Y Y Y Y Y Y N Y N Y Y Y
555 N Y N Y N N N N N N N N N Y N
13,000 Y N Y Y N N Y N Y N N Y N N N
1,000 Y N Y Y N N Y N Y N N N N N N
17,017 N N N N N Y N N N Y N Y N N N
123,456 Y Y Y N Y N Y N N N Y N N N Y
99

If these sums differ by a multiple of 11, then the number is divisible by 11. Since 0 = 0 × 11, in the example above, 7 differs from 7 by 0, or by a multiple of 11—so, 362,021 must be divisible by 11. Another example:



Since 22 is divisible by 11, so is 70,928.

A number is divisible by 12 if it meets the requirements for divisibility by 3 *and* by 4.

Thirteen is an "odd ball," very much like 7.

All even numbers divisible by 7 are also divisible by 14. All numbers divisible by 3 and 5 are divisible by 15.

Test for divisibility by 16 follows the pattern in 4 and 8. Since 10,000 is divisible by 16, then any number with a four-digit ending is divisible by 16.

Notes to accompany *Work Text* page 100

In columns A, B, C, D, at the top right of this page, the students complete the steps of a "Think of A Number" game. In column A the number is 5. In column B it is 8, and in column C it is 100. In column D, the students begin with any number they choose. (We have used 29 but, of course, any number would do as well.) Students can continue the game by making additional columns for other numbers thought of. They may find it instructive to consider the numbers 0, 1, 2, 2, 4 in order. In such a case the following observations could be made:

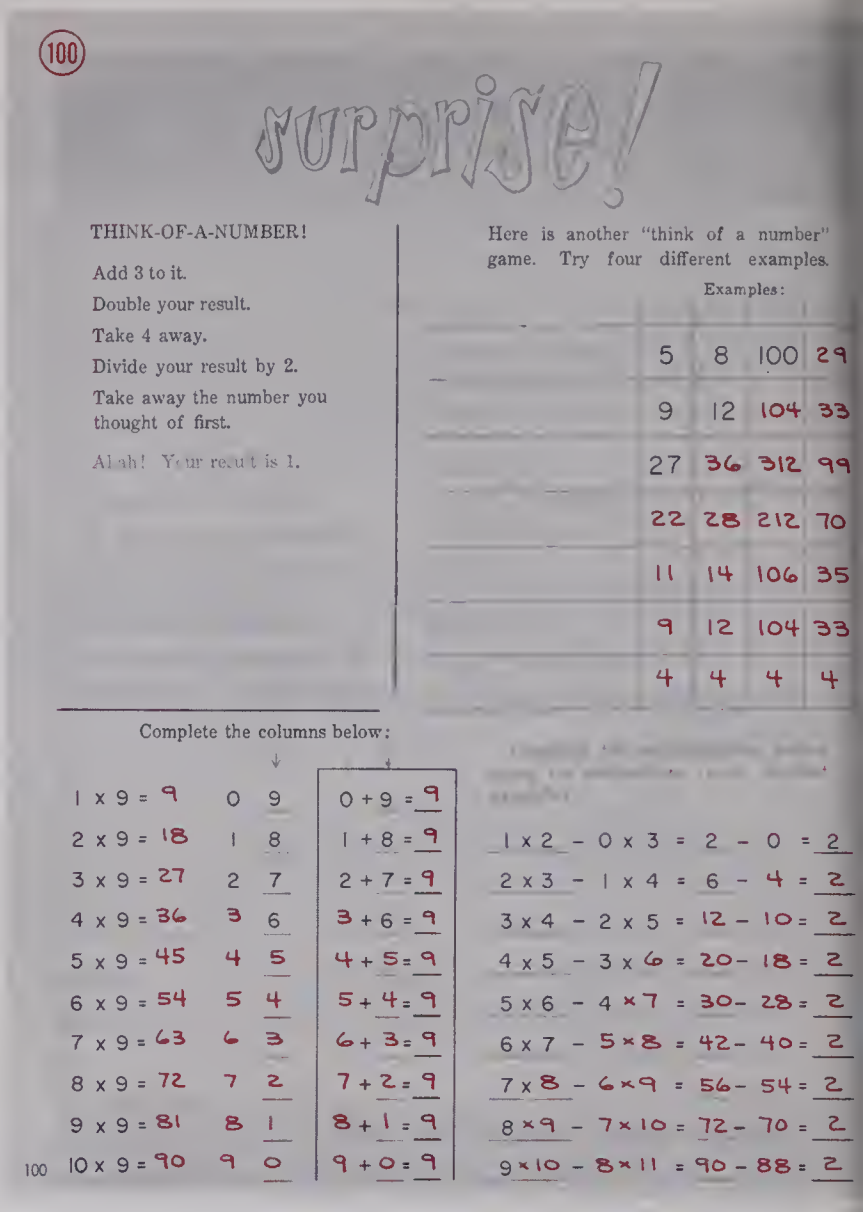
		Observation
Think of a number	0 1 2 3 4	numbers rise by steps of 1
Add 4	4 5 6 7 8	numbers rise by steps of 1
Multiply by 3	12 15 18 21 24	numbers rise by steps of 3
Take away number first thought of	12 14 16 18 20	numbers rise by steps of 2
Divide by 2	6 7 8 9 10	numbers rise by steps of 1
Take away 2	4 5 6 7 8	numbers rise by steps of 1
Take away number first thought of	4 4 4 4 4	all numbers the same

At the bottom left of the page, the students are asked to fill in the answers to the 9-times table. First they write the numbers of this table. Then they write separately the digits representing tens and units, so that 17 becomes *1 7*. The first entry, 9, represents 0 tens and 9 units, or 0 9. Students may observe that the numbers from 0 to 9 occur in order both in going down the tens column, and in coming up the units column.

Lastly, the students are to observe the result of adding the digits in each product. Students may ask, "No matter how high you go, will the digits always add up to 9?" Have them investigate for themselves. They will see that 99 becomes 9 + 9 = 18; 108 becomes 1 + 0 + 8 = 9; 999 becomes 9 + 9 + 9 = 27. If the sum of the digits is not 9, it is a multiple of 9.

Numbers divisible by 9 can be recognized by adding their digits. For example: 621 necomes 6 + 2 + 1 = 9; therefore 621 must be divisible by 9.

In the exercise at the bottom right, students will be surprised that the answer is always 2. At least, they will be if they carry out the instructions properly. It is important to stress the fact that multiplication and division should be carried out before addition and subtraction. (This is also important in algebra.) For example, 3 + 2 × 4 = 11. Mathematicians have made an agreement not to use 3 + 2 × 4 to mean *add 3 and 2, and then multiply by 4.* If this were intended, it would be shown as (3+2) × 4 = 5 × 4 = 20. Therefore, the expression 9 × 10 - 8 × 11 must be worked out as:


100
surprise!
THINK-OF-A-NUMBER!
Add 3 to it.
Double your result.
Take 4 away.
Divide your result by 2.
Take away the number you thought of first.
Aha! Your result is 1.
Here is another "think of a number" game. Try four different examples.
Examples:
5 8 100 29
9 12 104 33
27 36 312 99
22 28 212 70
11 14 106 35
9 12 104 33
4 4 4 4
Complete the columns below:
1 x 9 = 9 0 9 0+9=9
2 x 9 = 18 1 8 1+8=9
3 x 9 = 27 2 7 2+7=9
4 x 9 = 36 3 6 3+6=9
5 x 9 = 45 4 5 4+5=9
6 x 9 = 54 5 4 5+4=9
7 x 9 = 63 6 3 6+3=9
8 x 9 = 72 7 2 7+2=9
9 x 9 = 81 8 1 8+1=9
10 x 9 = 90 9 0 9+0=9
1 x 2 - 0 x 3 = 2 - 0 = 2
2 x 3 - 1 x 4 = 6 - 4 = 2
3 x 4 - 2 x 5 = 12 - 10 = 2
4 x 5 - 3 x 6 = 20 - 18 = 2
5 x 6 - 4 x 7 = 30 - 28 = 2
6 x 7 - 5 x 8 = 42 - 40 = 2
7 x 8 - 6 x 9 = 56 - 54 = 2
8 x 9 - 7 x 10 = 72 - 70 = 2
9 x 10 - 8 x 11 = 90 - 88 = 2

$9 \times 10 \quad - \quad 8 \times 11$

$90 \quad - \quad 88$

2

Students should notice that the same general pattern exists in each statement. Each involves, at the outset, four numbers that come within the natural order of counting. For example: 0, 1, 2, 3; 1, 2, 3, 4; 8, 9, 10, 11. In each case the middle numbers of the series are to be multiplied; likewise, the first and last (9×10 = 90 and 8×11 = 88). The difference between these pairs is always 2. Ask students what they think 42 × 43 − 41 × 44 should be. (Again, it is 2.) Have them work out the problem in order to check their guess. Next, have students fill in the spaces below in order to make true statements:

____ × ____ - 100 × 103 = 2

85 × 86 - ____ × ____ = 2

We do not explain in the *Work Text* why the answer is always 2, but the work on pages 112 through 115 studies and explains a similar *surprise.*

Notes to accompany *Work Text* page 101

On page 101 the first example deals with *square*

numbers; $1 \times 1, 2 \times 2, 3 \times 3$, etc. If these are written in numerical order, and each is subtracted from its successor, the result is a pattern:

1		4		9		16		25		36		49
	3		5		7		9		11		13	

The difference between any two of these square numbers is an odd number. If each odd number is subtracted from the next, the result is:

3		5		7		9		11		13		
	2		2		2		2		2			

In the second example, the numbers 0, 1, 2, 2, etc., are taken in numerical order and each is multiplied by *2 more than itself.* The answers are 0, 3, 8, 15, 24, 35, 48. If a scientist were confronted with these numbers, but did not know how they had been formed, he might wonder, "Is there a simple rule behind these numbers?" If he made a *table of the differences* between succeeding numbers, he would find:

0		3		8		15		24		35		48
	3		5		7		9		11		13	
		2		2		2		2		2		

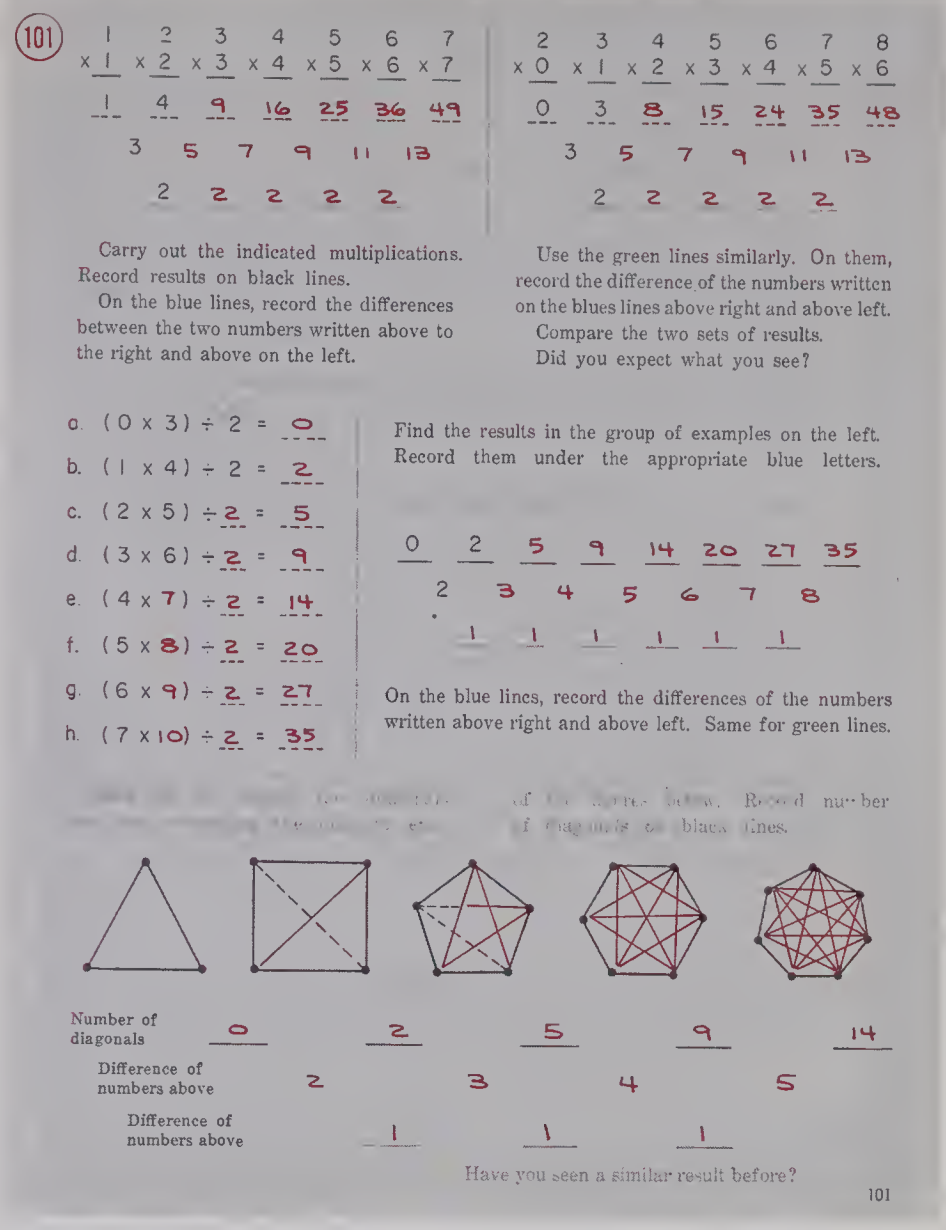
These numbers behave in a way that suggests a simple rule lies behind them. Of course, the scientist could not yet know the rule for certain. He could, however, predict that the next number in the middle row will be 15, and that that would mean the next number in the top row should be 63 and then test his prediction.

In the third example each consecutive number is multiplied *by 3 more than itself,* and the result is divided by 2. To test the resulting numbers: 0, 2, 5, 9, 14, 20, 27, 35, a table of the differences is made. There is an interesting reason for choosing this particular rule. The numbers 0, 2, 5, 9, 14, etc., can be related to the number of diagonals of a triangle, square, five-sided figure, six-sided figure, seven-sided figure, etc. By drawing the diagonals of these figures and counting them, the students are able to discover this relationship.

Notes to accompany *Work Text* page 102

The bags and counters introduced on this page will be used on *Work Text* page 103 to explain and invent tricks. The bag, containing an unknown number of counters, provides students with a way of picturing the number someone has thought of. This helps them to reason about it.

Game I is so simple that students do not really



101

1 2 3 4 5 6 7
×1 ×2 ×3 ×4 ×5 ×6 ×7
1 4 9 16 25 36 49
3 5 7 9 11 13
2 2 2 2 2

Carry out the indicated multiplications. Record results on black lines.

On the blue lines, record the differences between the two numbers written above to the right and above on the left.

2 3 4 5 6 7 8
×0 ×1 ×2 ×3 ×4 ×5 ×6
0 3 8 15 24 35 48
3 5 7 9 11 13
2 2 2 2 2

Use the green lines similarly. On them, record the difference of the numbers written on the blues lines above right and above left.

Compare the two sets of results.

Did you expect what you see?

a. (0 × 3) ÷ 2 = 0
b. (1 × 4) ÷ 2 = 2
c. (2 × 5) ÷ 2 = 5
d. (3 × 6) ÷ 2 = 9
e. (4 × 7) ÷ 2 = 14
f. (5 × 8) ÷ 2 = 20
g. (6 × 9) ÷ 2 = 27
h. (7 × 10) ÷ 2 = 35

Find the results in the group of examples on the left. Record them under the appropriate blue letters.

0 2 5 9 14 20 27 35
2 3 4 5 6 7 8
1 1 1 1 1 1

On the blue lines, record the differences of the numbers written above right and above left. Same for green lines.

Number of diagonals: 0 2 5 9 14

Difference of numbers above: 2 3 4 5

Difference of numbers above: 1 1 1

Have you seen a similar result before?

101

102

Anne gave each member of the class a bag and some counters. She said, "Think of a number. Put that number of counters in the bag, and tie it up.

"Put 3 counters by the bag. Now, if you tell me how many counters you have used, I will tell you what's in the bag."

	Betty	Jim	Ruth	Ted	Ben
Think of a number. Put that many counters in the bag.	5	3	9	14	0
Put 3 counters by the bag. How many counters all together?	8	6	12	17	3

Betty thought of 5; Jim thought of 3; Ruth thought of 9; Ted, 14; and Ben, zero.

"That's easy," Ben complained.

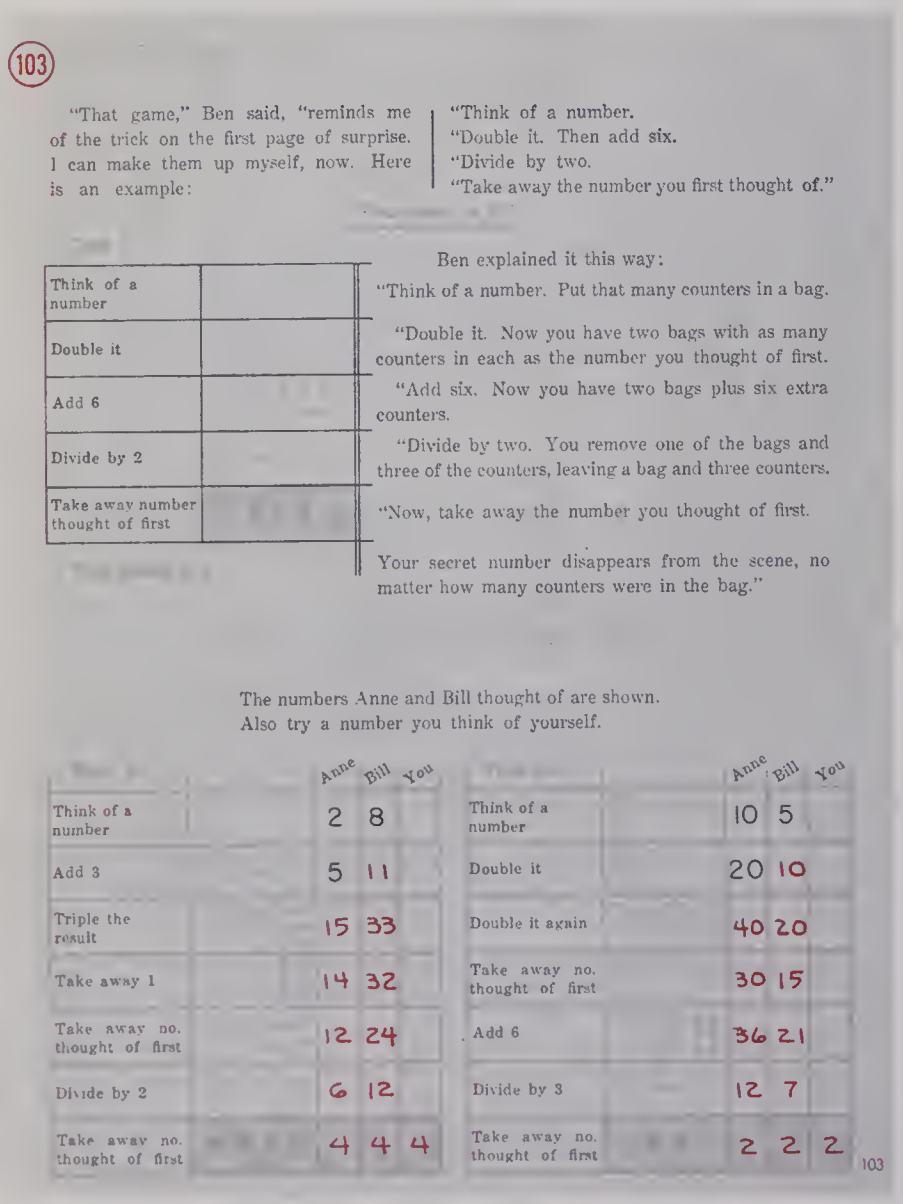
"All right," Anne continued, "you will each need extra bags for the next games."

Think of a number. Put that many counters in the bag.	4	11	15	17	0
Add 5. You now have the bag and extra counters.	9	16	20	22	5
Double it. You now have two bags and 10 extra counters. How many counters all together?	18	32	40	44	10
Think of a number	12	8	0	10	1
Double it	24	16	0	20	2
Add 6	30	22	6	26	8
Divide by 2. How many counters have you all together?	15	11	3	13	4
Think of a number	7	10	25	0	1
Add 7	14	17	32	7	8
Add the number you thought of first.	21	27	57	7	9
Subtract 5. How many have you all together?	16	22	52	2	4

102

need the bag picture. The answer can be found by working backwards. In Game III, students may reason, "Betty's answer is 15, so she must have had 30 counters before she divided by 2. She must have had 24 before she added 6; and before she doubled, she must have had 12." But by using the bag picture it is possible to reason, "Her answer was 15. A bag and 3 counters makes 15 counters. Therefore, the bag must contain 12 counters; 12 was her number." This is much quicker. *Game IV cannot be done at all by working backwards one step at a time,* but it is easy with bags. The students can argue, "Betty finished with 16. So two bags and two counters are 16 counters. Two bags must hold 14 counters; one bag must hold 7. That was Betty's number."

In Game II, the students may have different ideas about what is meant by *double it.* Some may think, *When you double a bag and 5 counters, you get 2 bags and 5 counters;* other may think the result is *a bag and 10 counters.* In both cases, the student would be doubling only a part of the number. In Betty's example, after adding 5 to her number, she had 4 counters in the bag and five beside the bag: 4 + 5. To double this, she did not say: 8 + 5; nor did she say: 4 + 10. She said: 8 + 10.



103

"That game," Ben said, "reminds me of the trick on the first page of surprise. I can make them up myself, now. Here is an example:

"Think of a number.
"Double it. Then add six.
"Divide by two.
"Take away the number you first thought of."

Ben explained it this way:

Think of a number		"Think of a number. Put that many counters in a bag.
Double it		"Double it. Now you have two bags with as many counters in each as the number you thought of first.
Add 6		"Add six. Now you have two bags plus six extra counters.
Divide by 2		"Divide by two. You remove one of the bags and three of the counters, leaving a bag and three counters.
Take away number thought of first		"Now, take away the number you thought of first.

Your secret number disappears from the scene, no matter how many counters were in the bag."

The numbers Anne and Bill thought of are shown.
Also try a number you think of yourself.

	Anne	Bill	You
Think of a number	2	8	
Add 3	5	11	
Triple the result	15	33	
Take away 1	14	32	
Take away no. thought of first	12	24	
Divide by 2	6	12	
Take away no. thought of first	4	4	4

	Anne	Bill	You
Think of a number	10	5	
Double it	20	10	
Double it again	40	20	
Take away no. thought of first	30	15	
Add 6	36	21	
Divide by 3	12	7	
Take away no. thought of first	2	2	2

103

Notes to accompany *Work Text* page 103

This page makes use of the bag and counter idea presented on page 102. Trick I is simply Game III of page 102, with a final step added. No new principle is involved in Tricks II and III.

The students will probably enjoy making up their own tricks. A trick can begin in *any way whatever.* It is always possible to find an ending that will make it work. The teacher may suggest a beginning, "Think of a number. Add 1, double. Subtract 1. Multiply by 3." There are 6 bags and 3 counters at this stage of the trick. The teacher may then challenge the class to finish it. A streamlined ending would be, "Add 9. Divide by 6. Take away the number first thought of. The answer is 2." (This can be checked by drawing the bags and counters.) But students rarely have sufficient experience to see this quick way home. They may give a much longer ending, with several subtractions and divisions. This does not matter; it may even make the trick more imposing and give more practice in arithmetic.

Tricks should be *invented* by drawing bags and counters, and *checked* by trying particular numbers. A trick has reached an ending only when no bag remains in the drawing—for the bag represents the unknown number chosen by the other person. So long as this unknown remains, we cannot predict the answer.

Notes to accompany *Work Text* page 104

There is no reason why a trick should not begin, "Think of a number. Multiply by 1000. Add 300." It would be wearisome, actually, to draw a thousand bags and three hundred stones. It is better to represent this by writing 1000 $\mathcal{B}$ + 300. All that the first column of this page is intended to do is establish this simple shorthand.

The second column introduces a very small change. We erase the top and bottom of the bag. This is simply a sneaky way of bringing in x (we use an italic "x" here and hereafter to represent the abbreviated bag-symbol explained on page 104) so that the work starts to look like high school algebra, instead of having the rather kindergarten air of bags and counters.

We try to keep the old meanings with the new sign. We may read $2x + 6$ on one occasion just as it is written, "two x plus six." On another occasion, we may read it as *two bags and six counters.*" Or again, we may read it as *twice the number thought of, with six added.*

Next day, Jerry said he had made up some of Ben's tricks with larger numbers.

"Instead of drawing counters, I'm going to use numbers to keep track of the extra counters."

(Below is the number Ben thought of. Include a number you think of.)

Trick IV

		Ben	You
Think of a number		19	
Add 50		69	
Double it		138	
Add 48		186	
Divide by 2		93	
Take away no. thought of first		74	

"Now," Jerry went on, "I'm going to have to draw large numbers of bags unless we can all agree that all I need to do is draw one bag and tell you how many I mean."

Trick V

		Ben	You
Think of a number		7	
Multiply by 10		70	
Add 130		200	
Divide by 5		40	
Divide by 2		20	
Take away no. thought of first		13	

"Those bags are hard to draw," Jerry complained. "So, I'm going to erase the top and bottom of the bag. All there is left is X which looks very much like the letter x. So, I will write 3 X + 6 for the 3 bags and six more."

Trick VI

		Ben	You
Think of a number		3	
Add 47		50	
Double it		100	
Take away 75		25	
Triple it		75	
Subtract number thought of first		72	
Add 18		90	
Divide by 5		18	
Take away no. thought of first		15	

Finish the picture and use Jerry's shorthand to show the following trick:

Trick VII

Words	Pictures	Shorthand
Think of a number		
Add 4		
Double it	8 :::: ::::	2X+8
Subtract 2	8 8 :::	2X+6
Add number thought of first	8 8 8 :::	3X+6
Divide by 3	8 ..	X+2
Take away no. thought of first	••	2

"If you show me the results of any line of the games or tricks, I'll tell you the number each player was thinking of to start with.

The only exception would be the last line of Ben's puzzle. There has to be at least one bag left in the picture."

Fred hid it. Can you discover his method?

Game II – page 102

Think of a number		4	11	15	7	0
3rd line →		18	32	40	24	10

Game III – page 102

Think of a number		12	8	0	10	1
3rd line →		30	22	6	26	8

Game IV – page 102

Think of a number		7	10	25	0	1
4th line →		16	22	52	2	4

Trick II – page 103

Think of a number		2	8
5th line		12	24

Trick III – page 103

Think of a number		10	5
5th line		36	21

Trick IV – page 104

Think of a number		19	
3rd line		138	

Trick V – page 104

Think of a number		7	
4th line		40	

Trick VI – page 104

Think of a number		3	
7th line		90	

Trick VII – page 104

Think of a number		4	9
5th line		18	33

Two Extras

Think of a number		1	7
		11	35

Think of a number		25	0
		225	150

Notes to accompany *Work Text* page 105

Fred's idea is not really new. Actually it was introduced on *Work Text* page 102. As a matter of fact, the third exercise—dealing with the fourth line of Game IV—has already been discussed in the notes for page 102. The first example on this page (concerning the third line of Game II) involves *2 bags and 10 counters.* If it is known that this represents 18 counters, then 2 bags must contain 8 counters, and each bag must contain 4 counters. So the number thought of was 4.

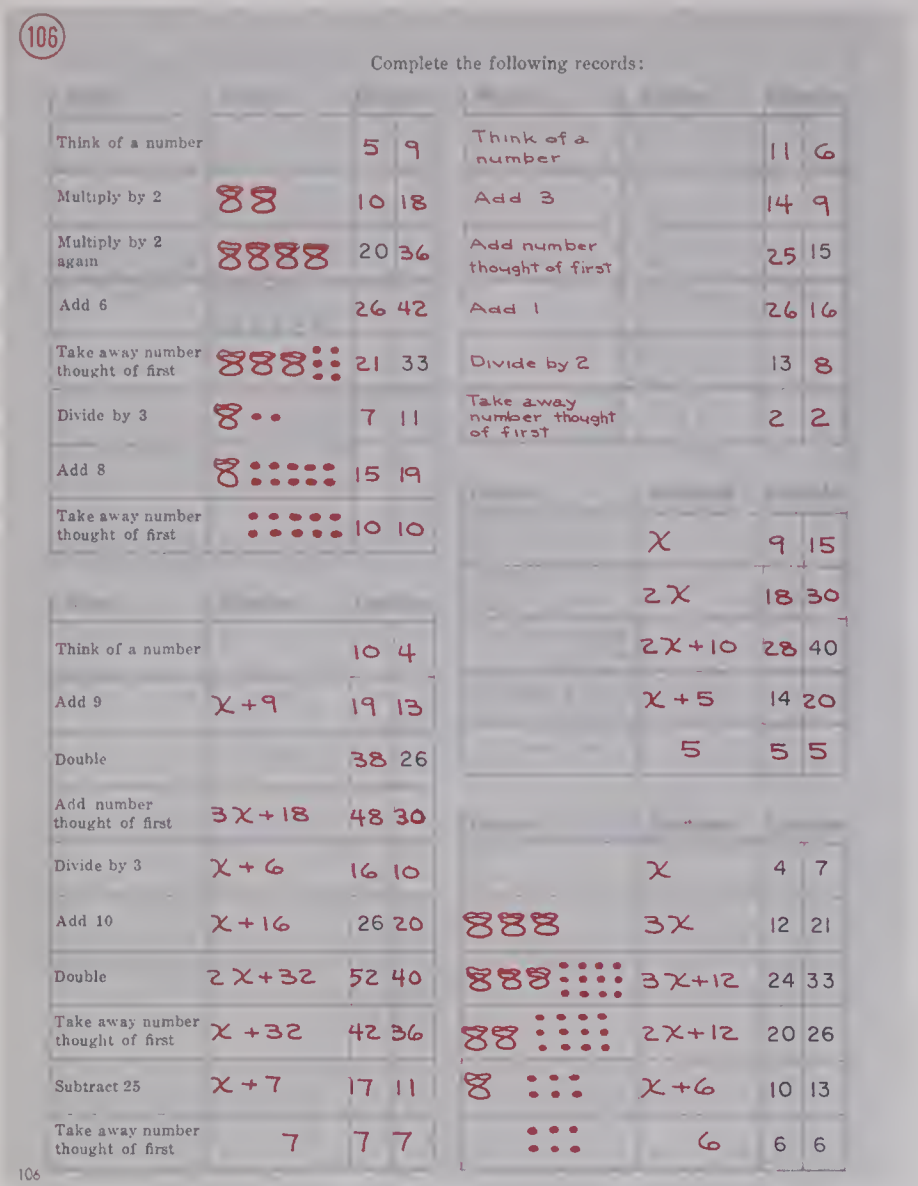
Lower down on the page, the shorter forms are used: 2 8 + 26 for *2 bags and 26 counters;* also $5x + 75$ for *5 bags and 75 counters.* Students may need to be reminded of the meaning behind the shorthand. The work then proceeds as before. If 5 bags and 75 counters make 90 counters then the 5 bags must contain altogether 15 counters (for you must add 15 to 75 to get 90). Each bag has 3.

In the last trick, the reasoning can proceed as follows: If 3 bags and 150 counters make 225 counters, we can begin by imagining the total 225 counters. We set 150 of them to one side; the remaining counters we divide equally into three bags. It is clear that 25 counters will go into each bag.

Notes to accompany *Work Text* page 106

The exercise at the bottom left of the page requires students to begin by translating from words to *x*-shorthand. (Students may want to turn the words into pictures first and then to use *x* as a sign for a bag.) Then, with regard to the first column, students may work back from 26 by reasoning: "A bag and 16 counters makes 26 counters, so the bag holds 10 counters," or "16 added to *x* makes 26, so *x* is 10," or "16 more than the number first thought of is 26, so the number must have been 10," whichever they prefer. (In the second column: $2x + 18 = 26$, therefore $2x = 8$ and $x = 4$.)

On the right-hand side of the page, students must first be able to work from pictures to words. Next, they must work from pictures to the *x*-shorthand. Finally, they are given the numbers used by two players, and the students must guess what the trick was. To work this final exercise, students have to know what two players do. If they were told what one player did, they could not be certain. For example, if they were told, "The player thought of 4. The next number he wrote was 12," they would not know whether the player had added 8 or multiplied by 3. But if students are told that one player goes from 4 to 12, and a second player goes from 7 to 21, *multiply by 3* fits both cases, but


106

Complete the following records:

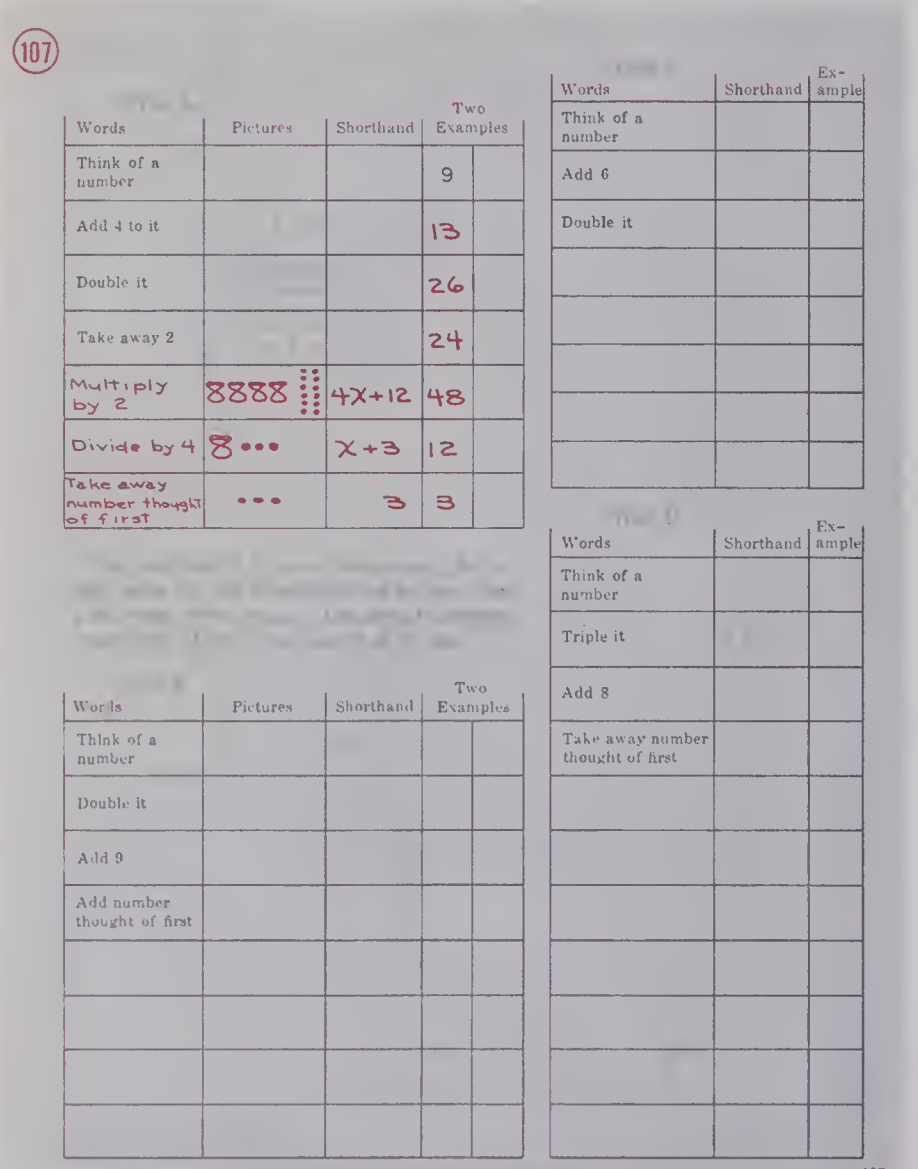
Think of a number		5	9
Multiply by 2	88	10	18
Multiply by 2 again	8888	20	36
Add 6		26	42
Take away number thought of first	888 ⁘	21	33
Divide by 3	8 ••	7	11
Add 8	8 ⁘	15	19
Take away number thought of first	⁘	10	10

Think of a number		11	6
Add 3		14	9
Add number thought of first		25	15
Add 1		26	16
Divide by 2		13	8
Take away number thought of first		2	2

Think of a number		10	4
Add 9	X+9	19	13
Double		38	26
Add number thought of first	3X+18	48	30
Divide by 3	X+6	16	10
Add 10	X+16	26	20
Double	2X+32	52	40
Take away number thought of first	X+32	42	36
Subtract 25	X+7	17	11
Take away number thought of first	7	7	7

	X	9	15
	2X	18	30
	2X+10	28	40
	X+5	14	20
	5	5	5

	X	4	7
888	3X	12	21
888 ⁘	3X+12	24	33
88 ⁘	2X+12	20	26
8 ⁘	X+6	10	13
⁘	6	6	6


107

Words	Pictures	Shorthand	Two Examples	
Think of a number			9	
Add 4 to it			13	
Double it			26	
Take away 2			24	
Multiply by 2	8888 ⁘	4X+12	48	
Divide by 4	8 •••	X+3	12	
Take away number thought of first	•••	3	3	

Words	Shorthand	Example
Think of a number		
Add 6		
Double it		

Words	Pictures	Shorthand	Two Examples	
Think of a number				
Double it				
Add 9				
Add number thought of first				

Words	Shorthand	Example
Think of a number		
Triple it		
Add 8		
Take away number thought of first		

107

add 8 does not. In the next step, going from 12 to 24 might be *double,* but this does not send 21 to 33. *Add 12* fits both players. The next step, *take away number first thought of,* is perhaps not easy to guess. Then comes *divide by 2,* and finally *take away number first thought of.*

Notes to accompany *Work Text* page 107

On page 107 students are asked to make up their own endings to tricks. It has already been noted on page 103 that to end a trick quickly requires considerable foresight and some experience. Therefore, it may not be wise to ask students to finish the trick in 2 or 3 steps. Rather, be pleased if they can finish it at all.

To reach a quick ending, students must learn to realize several things: first, that a trick is complete when it reaches a stage where *no bag* is drawn. Second, that the step before this is usually, *take away the number first thought of* (that is, *take away a bag).* Thirdly, that there must be *one bag* before this step. Fourthly, that it is usually possible to get *one* bag by dividing.

In the first example on page 107 there are 2 bags and 6 counters at the end of the fourth step. A streamlined ending would be, *divide by 2.* This would leave 1 bag and 3 counters. The next step would be the standard ending, "Take away the number first thought of." This would leave no bag in the picture; only 3 counters. The answer is 3, and the trick has been completed.

Sometimes another step is necessary in order to make division easier. For example, if a trick has reached a stage where there are 2 bags and 7 counters, it is not the best time to divide by 2, because it would result in a fraction: 3 1/2 counters. To keep the trick simple, the following ending would be preferable:

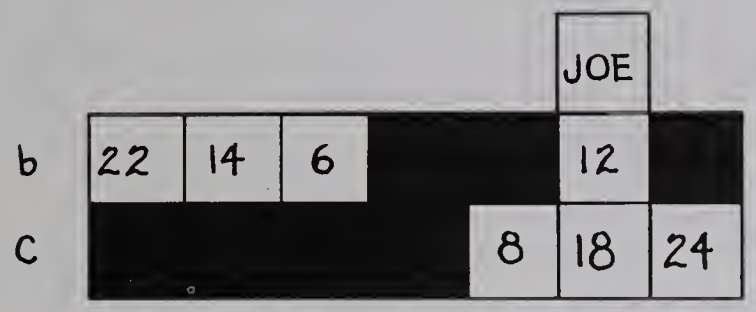
Instruction	*Situation*	*Reason for Step*
	$2x + 7$	
Add 1	$2x + 8$	Make division by 2 easy
Divide by 2	$x + 4$	To get *one* bag
Take away number first thought of	4	To get *no* bag

Specific notes are not considered necessary for *Work Text* page 108.

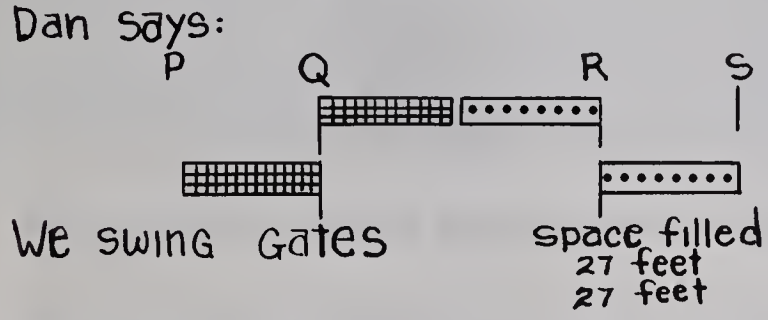
Notes to accompany *Work Text* page 109

On this page, Fred explains the reasoning for each step. *Students may want to first see if they can complete the table at the top of the page, before reading Fred's explanation.*

In an ordinary crossword puzzle, a word that may be very hard to guess at first, may become obvious when other words around it have been filled in. The same is true of these number tricks. Look at Jane's work, for example. She has nothing at all in the first three rows. It is necessary to work back from Jane's fourth row, (row d) by using information provided by the other students in the chart. Look at rows *b* and *c:*



The only link between these rows is given by Joe's 12 and 18. This suggests *add 6,* but could it perhaps be, *multiply by 1 1/2*"? Obviously, we need more information. If we study rows a and b for Ann, Ben and Mat, we learn that row *b* is double row *a.* This means that rows *b* and *c* now look like this:



It is now possible to conclude that *add 6* is the step from *b* to *c.*

Notes to accompany *Work Text* page 110

Skeleton solutions to the problems on page 110 are indicated below. The rows are referred to by letters: *a, b, c, d,* etc.

Problem 1

Step	*Evidence*	*Results*
a to *b*	Jane, Joe	*Add 3.* Fill in Anne.
b to *c*	Anne, Mat	*Double.* Fill in Ben, Jane, Ruth and Joe, row c.
c to *d*	Ben, Ruth	*Add 4.* Fill in Anne, Mat, Jane, Joe, row *d.*
d to *e*	Ben, Jane	*Divide by 2.* Fill in row *e.* Work back for spaces under Ben, Mat, Ruth and Betty.
e to *f*	Ben, Betty	*Take away number first thought of.* Fill in row *f.*

Problem 2

		The top row is nearly all blank! We must look lower down.
b to *c*	Anne, Ben, Mat, Ruth	*Add 6.* So in row *b,* shorthand must read 3*x.* Also, in row a, shorthand reads *x,* for *a bag.*

"If you show me part of a record of the kind of trick and games we've been playing, I can often figure out the rest. You don't have to show me the words or pictures or shorthand.

"Here is a sample of the ways two tricks might get started. I'll explain my idea as we go along."

Words	Pictures	Shorthand	Anne	Ben	Mat	Jane	Ruth	Joe	Betty
Think of a number	8	X	5	3	1	11	7	0	25
Add 5	8•••••	X + 5	10	8	6	16	12	5	30

"Consider Anne's results. She thought of 5. By applying some rule not shown, she was led to 5. The rule might be 'double it' or, perhaps, 'add 5.' Ben thought of 3. 'Double it' would lead to 6. But his result is 8. However, 'add 5' would explain the results of both Anne and Ben.

"The rule 'add 5' would also explain the results recorded for Jane and Joe. Applying the rule to the numbers Mat and Betty thought of, they would lead to 6 and 30. And we can see that the number Ruth thought of must have been 7, because 7 + 5 = 12.

"Now we can draw the pictures and show the idea in shorthand."

Words	Pictures	Shorthand	Anne	Ben	Mat	Jane	Ruth	Joe	Betty
Think of a number	8	X	0	100			10	12	25
Triple it	888	3X	0		21	33	30		75
				315				51	90

"From Anne's results in the first 2 rows, I know the rule must be 'add 0' or it must involve multiplication. Ruth thought of 10 and the rule led her to 30. It must be 'multiply by three' or 'triple the number you thought of.'

"That would also explain Betty's results. Now I can draw pictures and write the shorthand for the second row. I can also find out the missing items in the first and second lines."

Next, consider the second and third rows.

Words	Pictures	Shorthand	Anne	Ben	Mat	Jane	Ruth	Joe	Betty
Think of a number	8	X	0	100			10	12	25
			0		21	33	30		75
Add 15	888:::::	3X+15	15	315	36	48	45	51	90

"Ben's results suggest right away that the rule might be 'add 15.' That rule explains the results given for Joe and Betty. Now I can complete the third line."

Words	Pictures	Shorthand	Anne	Ben	Mat	Jane	Ruth	Joe	Betty
a. Think of a number	8	X	11	7	3	8	1	6	9
b. Double it	88	2X	22	14	6	16	2	12	18
c. Add 6	88:::	2X+6	28	20	12	22	8	18	24
d. Divide by 2	8•••	X+3	14	10	6	11	4	9	12
e. Subtract number thought of first	•••	3	3	3	3	3	3	3	3

"Consider lines a and b," Fred advises.

a. Think of a number	8	X	11	7	3	8	1	6	9
b. Double	88	2X	22	14	6	16	2	12	18

"In line a, I begin by drawing a bag. When anyone thinks of a number, I suppose they put that many counters in the bag. Then I use Jerry's shorthand. I erase the top and bottom of the bag. The part that is left looks like the letter 'X.'

"Next, I consider the results shown for Anne, Ben, and Mat. The rule must be 'double' the number you thought of. So, I write this idea in words, pictures, and shorthand. Now, I can fill in the missing results for Ruth, Joe, and Betty."

Results of line b			22	14	6	16		12	
c. Add 6	88:::	2X+6	28	20	12	22	8	18	24

By studying the results for Ruth, Joe and Betty, I know the rule must be 'add 6.'

Results of line c						22	8	18	24
d. Divide by 2	8•••	X+3	14	10	6	11	4	9	12

By studying the results shown for Anne, Mat and Ruth, I know the rule is 'divide by 2.'

a. Number thought of first			11	7	3	8	1		9
d. Results of line e			14		6	11	4		
e. Take away number thought of first	•••	3	3	3	3	3	3	3	3

"Consider the lines a, d, and e. The results for Ben, Joe and Betty give the clue. The rule must be: from the results in line d, 'take away the number you thought of first.' Now, you can complete reconstruction of the entire record."

a to *b*	Shorthand	*Multiply by 3* for this changes *x, a bag,* into 3*x, three bags.* Fill in row a—Anne, Ben, Mat, Ruth.
c to *d*	Ben, Mat, Ruth	*Take away number first thought of.* Write $2x + 6$ in row d, shorthand.
e to *f*	Words, row *f*	We are told *subtract the number you thought of first.* So there must have been $x + 3$ in row e, shorthand. Write it in.
d to *e*	Shorthand	*Divide by 2* for this changes $2x + 6$ into $x + 3$. Rest can now be filled in.

Problem 3

The picture for row *c* shows a bag and 4 counters; the picture for row *a* shows a bag. So the numbers in row *c* must be 4 more than the numbers in row *a*. We can now fill in row *a* and row *c*. We continue with this:

Step	*Evidence*	*Results*
a to *b*	Columns B, C	*Add 10.* Fill in row *b.*
b to *c*	Columns A, B, C, D	*Subtract 6.*
c to *d*	Columns A, C	*Subtract number first thought of.*

Problem 4

Row *c* has shorthand $2x + 20$, which means *twice the number thought of, and 20.* So F thought of 0, and G thought of 3. Fill this in.

Columns F and H suggest that we get from row *c* to *d* by saying, *subtract the number first thought of.* So the shorthand in row *d* should read $x + 20$. As row d has 26 under E, this means E must have begun with 6. Row *d* can be completed by putting 23 under column G.

From row *b* to row *c,* only column F tells us anything. It suggests, *add 20.* So we can enter 6 and 14 in row *b,* under G and H, for these are 20 less than the numbers in row *c.*

Only column G helps with the step from *d* to *e.* It suggests *subtract 18.* If we try this, and end up with *subtract the number first thought of,* we find this works.

Problem 5

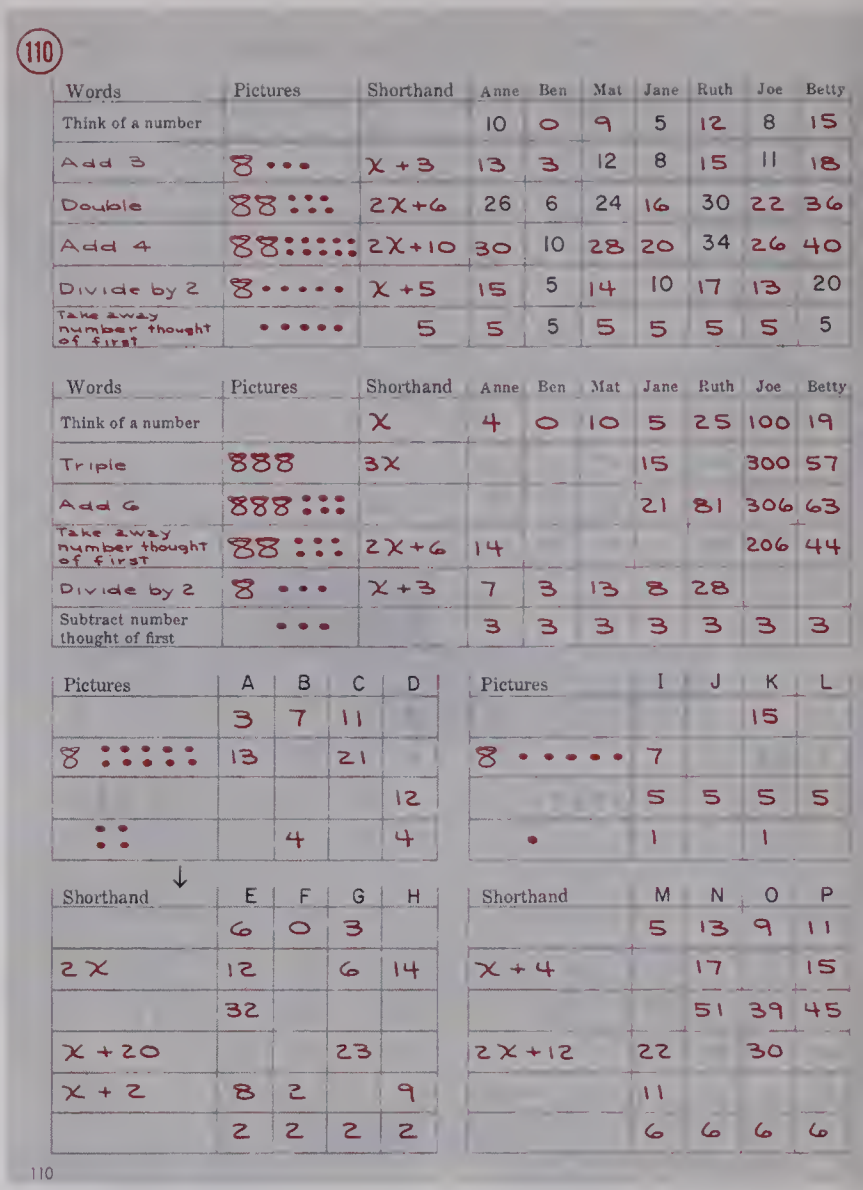
Step	*Evidence*	*Results*
a to *b*	Columns J, K, L Row *c,* picture	*Add 5.* Fill in row *b.* Fill in 5 all along row *c.*
b to *c*	Columns I, J, K, L	*Subtract number first thought of.*
c to *d*	Columns I, K	We see that 5 goes to 1. As row *c* is all fives, row *d* must be all ones. We cannot tell whether instruction is *subtract 4,* or *divide by 5.*

Problem 6

In the bottom row, the shorthand says 6. This means, "The answer is 6." So we can write 6 all the way along row *f.* In row *e,* the shorthand says $x + 6$, *a bag and 6 counters.* So, N put 13 counters in his bag, 0 put 9, P put 11. We enter these numbers in row *a.* In row *c,* the shorthand reads $3x + 12$, *three bags and 12 counters.* So we can put 51; 39; 45 in row *c* under N, O, P. Also, we can see from the 27 in row *c* that M must have had 5 counters in his bag. Write 5 under M, in row *a.*

Columns M and O show that the step from row *a* to row *b* is *add 4.* We fill in row *b.* We now see that row *c* has numbers 3 times those in row *b.* (We might also see that $3x + 12$ is 3 times $x + 4$.)

Columns N and P suggest that the step from row *c* to row *d* is *subtract the number first thought of.* From row *d* to row *e,* the step is *divide by 2.*


110
Words Pictures Shorthand Anne Ben Mat Jane Ruth Joe Betty
Think of a number 10 0 9 5 12 8 15
Add 3 8 ••• X + 3 13 3 12 8 15 11 18
Double 88 ::: 2X+6 26 6 24 16 30 22 36
Add 4 88::::: 2X+10 30 10 28 20 34 26 40
Divide by 2 8••••• X + 5 15 5 14 10 17 13 20
Take away number thought of first ••••• 5 5 5 5 5 5 5 5
Words Pictures Shorthand Anne Ben Mat Jane Ruth Joe Betty
Think of a number X 4 0 10 5 25 100 19
Triple 888 3X 15 300 57
Add 6 888::: 21 81 306 63
Take away number thought of first 88 ::: 2X+6 14 206 44
Divide by 2 8 ••• X + 3 7 3 13 8 28
Subtract number thought of first ••• 3 3 3 3 3 3 3
Pictures A B C D
3 7 11
8 ::::: 13 21
12
:: 4 4
Pictures I J K L
15
8 ••••• 7
5 5 5 5
• 1 1
Shorthand E F G H
6 0 3
2X 12 6 14
32
X + 20 23
X + 2 8 2 9
2 2 2 2
Shorthand M N O P
5 13 9 11
X + 4 17 15
51 39 45
2X + 12 22 30
11
6 6 6 6

Notes to accompany *Work Text* page 111

There are ten rows of examples on page 111. For example, in 1A the problem might be stated this way: "I think of a number. I add 8 to it. I get 18. What number was I thinking of?" In example 2F: "Two bags and 5 counters make a total of 9 counters. How many counters are in each bag?" Students can answer such questions by using their common sense. *No formal rules of algebra should be taught.*

Example 3C could be written as follows:

$$(4\times\_\_\_\_) - 3 = 17$$

If $4x - 3$ indicates 17 counters, before subtracting the 3 counters, there must have been 20. If $4x$ indicates 20 counters, x or bag must mean 5.

Any of the approaches illustrated above can be applied to most of the questions on this page. In the eighth row of examples it is probably easiest to argue that $9 + 3x + 5x$ means 9 counters and 3 bags and 5 bags, which is the same as 9 counters and 8 bags. In the ninth row of examples, to 4 bags we add 7 counters and then remove 2 bags. This is the same as saying 2 bags and 7 counters, or $2x + 7$. In the tenth row of examples, we are told what $9 + 2x + 5x$ (or $9+7x$) is, and we have to discover what x is.

11 Fred repeats his claim:

"Some of these guessing games are becoming more complicated. But, if you give me the shorthand and results in any row, I can still tell the number each player thought of first."

	Shorthand	A	B	C	D	E	F	G	H	I
One row of results	$X+8$	18	19	20	8	100	16	39	29	9
Number each player thought of first	X	10	11	12	0	92	8	31	21	1
One row of results	$2X+5$	15	35	55	5	7	9	115	91	67
Number thought of first	X	5	15	25	0	1	2	55	43	31
One row of results	$4X-3$	1	9	17	97	37	397	25	41	57
Number thought of first	X	1	3	5	25	10	100	7	11	15
One row of results	$10+X$	12	20	11	60	10	21	39	35	17
Number thought of first	X	2	10	1	50	0	11	29	25	7
One row of results	$3X-6$	15	0	30	24	9	144	36	69	84
Number thought of first	X	7	2	12	10	5	50	14	25	30
One row of results	$200-X$	195	199	125	200	193	190	50	100	19
Number thought of first	X	5	1	75	0	7	10	150	100	181
One row of results	$25X+25$	225	125	50	275	150	25	400	200	250
Number thought of first	X	8	4	1	10	5	0	15	7	9
One row of results	$9+3X+5X$	9	33	17	65	25	49	97	89	169
Number thought of first	X	0	3	1	7	2	5	11	10	20
One row of results	$4X+7-2X$	11	7	29	9	21	27	107	57	17
Number thought of first	X	2	0	11	1	7	10	50	25	5
One row of results	$9+2X+5X$	65	44	16	58	9	72	86	184	79
Number thought of first	X	8	5	1	7	0	9	11	25	10

112

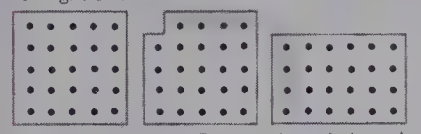
"If I think of any number, I can always get the same results in two ways that seem very different.

"I'll keep track of these ways in shorthand — but I'm not going to use bags or 'X' because 'n' or any other mark would be just as good, as long as we agree it means the number you thought of first."

(Try some examples of your own)

Words	Shorthand	Ben	Al	Mary	Jane	A	B	C	D
Think of a number		3	5	11	4	10	6	9	8
Multiply your number by itself		9	25	121	16	100	36	81	64
1 less than that		8	24	120	15	99	35	80	63
1 more than your number	$n+1$	4	6	12	5	11	7	10	9
1 less than your number	$n-1$	2	4	10	3	9	5	8	7
Multiply these together	$(n+1)\cdot(n-1)$	8	24	120	15	99	35	80	63

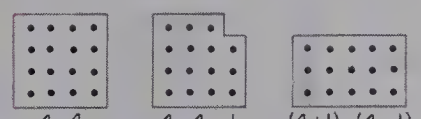
"You can make a picture to show why this happens. For example, the number Al thought of was 5."



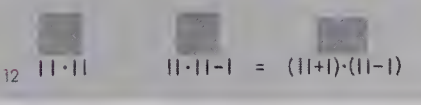
$5\cdot5$ $5\cdot5-1$ $(5+1)\cdot(5-1)$

"After you take one dot away from 5 rows of 5 dots, you have just enough to make 4 rows of 6 dots.

"Jane's example works the same way."



$4\cdot4$ $4\cdot4-1$ $(4+1)\cdot(4-1)$



$11\cdot11$ $11\cdot11-1 = (11+1)\cdot(11-1)$

"This is useful as a short cut. When I want to know the answer to 21×19, I think of this as $(20+1)\cdot(20-1)$; and I know that I can find the result by finding $20\cdot20-1$. It is 399. You can use this same method in the following:

$31 \times 29 =$ 899 $51 \times 49 =$ 2499

$13 \times 11 =$ 143 $101 \times 99 =$ 9999

$41 \times 39 =$ 1599 $71 \times 69 =$ 4899

"It is also a short cut when I want to find one less than 21×21. I know that $21\cdot21-1 = 21-1 \times 21-1$ or 22×20. The result is 440."

$31\cdot31-1 =$ 960 $51\cdot51-1 =$ 2600

$41\cdot41-1 =$ 1680 $101\cdot101-1 =$ 10200

"So, of course, I can use this short cut in the following:"

$31\cdot31 =$ 961 $51\cdot51 =$ 2601

$21\cdot21 =$ 441 $41\cdot41 =$ 1681

Notes to accompany *Work Text* page 112

Page 112 is based on the following "coincidence" in arithmetic: think of any three consecutive numbers —say 7, 8, and 9. If the first and the last in the sequence are multiplied ($7\times9=63$), and then if the middle one is multiplied by itself ($8\times8=64$), the result in the first instance is always 1 less than the result in the second.

On this page we do four things:
1) We try various numbers to see if they confirm the above statement.
2) We draw pictures to show why it happens.
3) We use the result as a "shortcut" for mental arithmetic.
4) We translate the idea into shorthand. The following is a brief summary of what is involved in each of these steps:

1) This involves straightforward arithmetic.
2) Rectangles are used to illustrate multiplication. If 8×8 is 1 more than 9×7, we ought to be able to arrange 8 rows with 8 counters in each, remove one counter, and rearrange the rest to make 9 rows of 7 counters. The pictures are supposed to suggest this procedure.
3) Involves only simple arithmetic.
4) As in the first example, we bring in n as an abbreviation for the middle number of a series. It would be 8 in the example above of 7, 8, 9. The students may think of 9 as *the number after 8,* but this expression does not lend itself to the shorthand of algebra. Algebra tries to keep the signs such as + and −; it does not have any sign for "after." What do we do to 8 to get 9? We add 1. So n + 1 expresses the number that follows n in the order of counting. In the same way, n-1 expresses the number that precedes n in the counting order.

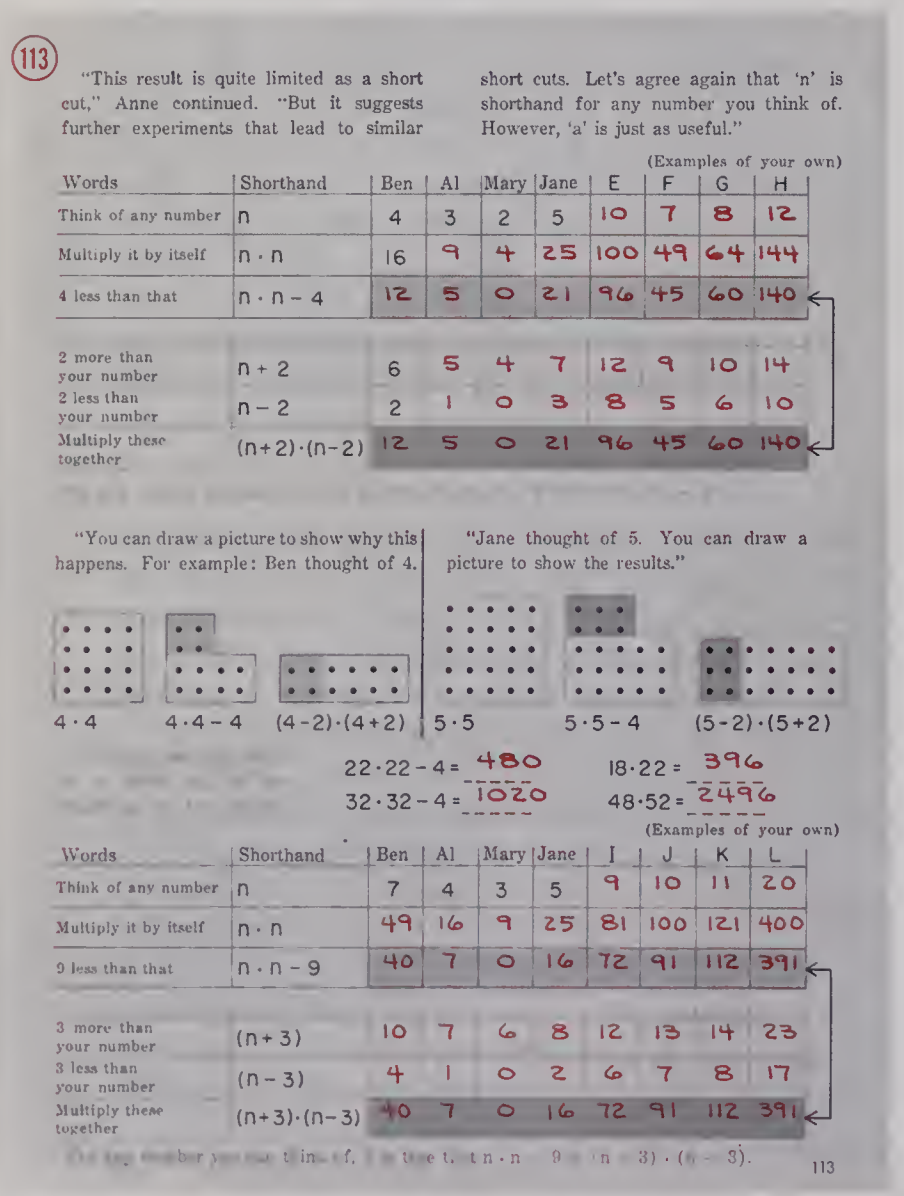
Writing multiplication is one of the irritating things in algebra. First of all, we are unable to use the familiar multiplication sign ($\times$) of arithmetic, because it is so liable to be mistaken for the letter x. Sometimes we use no sign at all: $7n$ stands for *seven bags* or *seven times the number first thought of.* This resembles the way we speak. *Seven fives* expresses multiplication; it means the same as *seven times five.* In algebra it is perfectly acceptable to write nn for *the number thought of multiplied by itself.* Just as we might say *five fives* and *eight eights,* we also say *en ens.* Sometimes, however, a dot is used to remind the reader that multiplication is intended. It is possible to read n·n as *en times en* or *en ens.* To help the students remember that the

dot means multiplication, you can start with a dot that resembles a little Sputnik: ✕ . (This is not likely to be mistaken for *n*.) Gradually, you can make the lines shorter and shorter, and end with simply a dot.

The order of operations is important. In examples like: $n \cdot n - 1$ or $5 \cdot 5 - 1$, multiplication must always be done before addition or subtraction is done. $5 \cdot 5 - 1$ means (1) find five fives; (2) *then* subtract 1. $(25 - 1 = 24)$

Brackets would have to be used if subtraction were to be done first. $5 \cdot (5 - 1)$ means (1) take 1 from 5: $5 - 1 = 4$; (2) *then* multiply by 5: $4 \times 5 = 20$. Notice that the brackets in $(n+1) \cdot (n-1)$ give priority to addition and subtraction. They say, in effect, "choose a number, and add 1 to it. Write down the results. Next subtract one from your chosen number, and write that down. *Now* multiply the two numbers you have written down."

Children may be amused by a mnemonic device: the signs ( ) are like blinders on a horse. Until you have worked out what is between them, you cannot see anything in the world outside them! You can hold your hands, like parentheses, at the side of your eyes, to show that you see, on the blackboard, only what is between the parentheses.



113

"This result is quite limited as a short cut," Anne continued. "But it suggests further experiments that lead to similar short cuts. Let's agree again that 'n' is shorthand for any number you think of. However, 'a' is just as useful."

(Examples of your own)

Words	Shorthand	Ben	Al	Mary	Jane	E	F	G	H
Think of any number	n	4	3	2	5	10	7	8	12
Multiply it by itself	n · n	16	9	4	25	100	49	64	144
4 less than that	n · n − 4	12	5	0	21	96	45	60	140
2 more than your number	n + 2	6	5	4	7	12	9	10	14
2 less than your number	n − 2	2	1	0	3	8	5	6	10
Multiply these together	(n+2)·(n−2)	12	5	0	21	96	45	60	140

"You can draw a picture to show why this happens. For example: Ben thought of 4.

4 · 4 | 4 · 4 − 4 | (4 − 2) · (4 + 2)

"Jane thought of 5. You can draw a picture to show the results."

5 · 5 | 5 · 5 − 4 | (5 − 2) · (5 + 2)

$22 \cdot 22 - 4 =$ 480 $\quad 18 \cdot 22 =$ 396

$32 \cdot 32 - 4 =$ 1020 $\quad 48 \cdot 52 =$ 2496

(Examples of your own)

Words	Shorthand	Ben	Al	Mary	Jane	I	J	K	L
Think of any number	n	7	4	3	5	9	10	11	20
Multiply it by itself	n · n	49	16	9	25	81	100	121	400
9 less than that	n · n − 9	40	7	0	16	72	91	112	391
3 more than your number	(n+3)	10	7	6	8	12	13	14	23
3 less than your number	(n−3)	4	1	0	2	6	7	8	17
Multiply these together	(n+3)·(n−3)	40	7	0	16	72	91	112	391

Or, if the number you start in is *n*, the result is $n \cdot n - 9 = (n + 3) \cdot (n - 3)$.

113

Notes to accompany *Work Text* page 113

On page 112 the following scheme was introduced:

Any number	One more		One less	Difference
$5 \times 5 = 25$	6	×	4 = 24	1
$8 \times 8 = 64$	9	×	7 = 63	1
$10 \times 10 = 100$	11	×	9 = 99	1

Someone may ask, "Why *one* more and *one* less? What would happen if you took *two* more and *two* less?" This is what we experiment with on *Work Text* page 113.

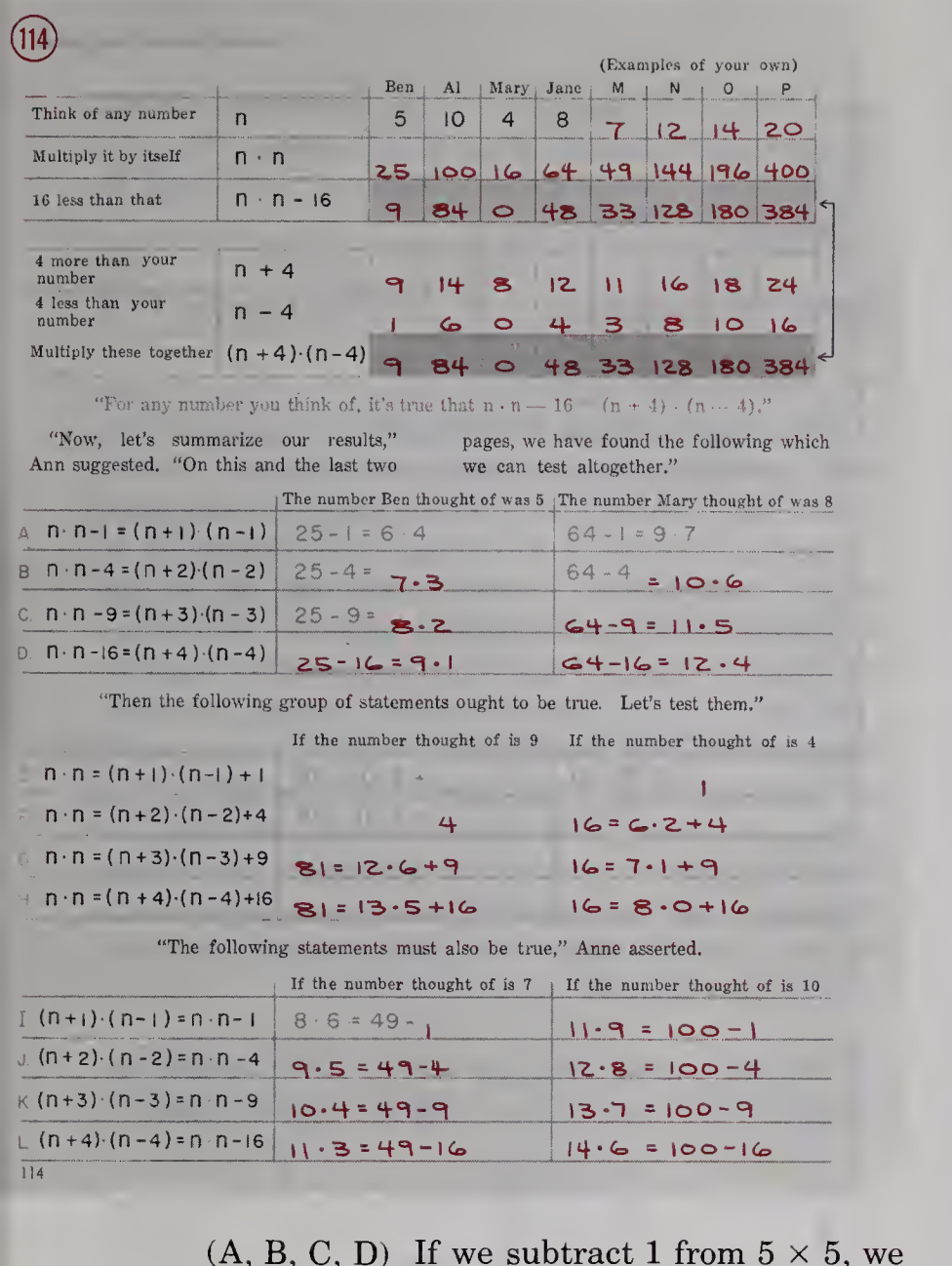
Any number	Two more		Two less	Difference
$5 \times 5 = 25$	7	×	3 = 21	4
$8 \times 8 = 64$	10	×	6 = 60	4
$10 \times 10 = 100$	12	×	8 = 96	4

As on page 112, we draw pictures; we use the general statement as a shortcut for mental arithmetic; and we express the statement in shorthand. Any of the points mentioned on page 112 may prove valuable here. The next question is clearly, "What about *three more* and *three less*?" That, too, is studied on this page. Your students may easily guess that on page 114 they will be asked to consider the possibility of *four more* and *four less*. Perhaps they can carry on the investigation for themselves, *before they read page 114*. If so, *excellent*.

Notes to accompany *Work Text* page 114

On this page, after an investigation of *four more and four less*, the students are presented with a summary of the four general patterns they have been following. These are set out in rows A, B, C, D, together with numerical examples that demonstrate what the shorthand means. In E, F, G, H, the same facts are set out in a slightly different way; instead of saying that 9×7 is 1 less than 8×8, we say that 8×8 is 1 more than 9×7. In rows I, J, K, L, we repeat the shorthand of A, B, C, D, but we reverse the order: instead of saying $25 - 1 = 6 \times 4$, we say that $6 \times 4 = 25 - 1$.

In other words, the twelve rows, A through L, assemble the facts already known, and state the same facts in three different forms:



(114)

(Examples of your own)

		Ben	Al	Mary	Jane	M	N	O	P
Think of any number	n	5	10	4	8	7	12	14	20
Multiply it by itself	n · n	25	100	16	64	49	144	196	400
16 less than that	n · n − 16	9	84	0	48	33	128	180	384
4 more than your number	n + 4	9	14	8	12	11	16	18	24
4 less than your number	n − 4	1	6	0	4	3	8	10	16
Multiply these together	(n + 4)·(n − 4)	9	84	0	48	33	128	180	384

"For any number you think of, it's true that n · n — 16 (n + 4) · (n — 4)."

"Now, let's summarize our results," Ann suggested. "On this and the last two pages, we have found the following which we can test altogether."

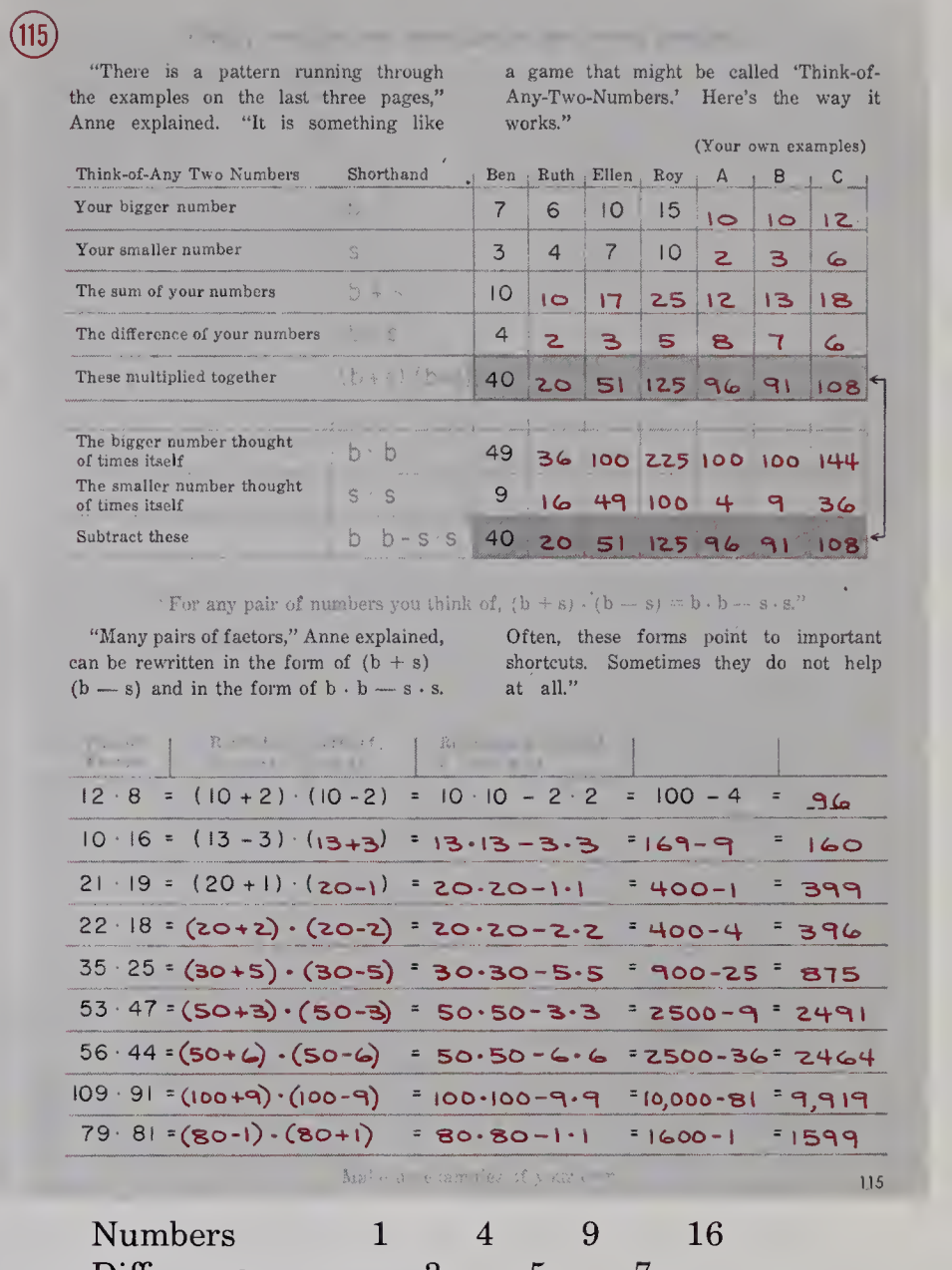
		The number Ben thought of was 5	The number Mary thought of was 8
A	n · n − 1 = (n + 1) · (n − 1)	25 − 1 = 6 · 4	64 − 1 = 9 · 7
B	n · n − 4 = (n + 2) · (n − 2)	25 − 4 = 7 · 3	64 − 4 = 10 · 6
C	n · n − 9 = (n + 3) · (n − 3)	25 − 9 = 8 · 2	64 − 9 = 11 · 5
D	n · n − 16 = (n + 4) · (n − 4)	25 − 16 = 9 · 1	64 − 16 = 12 · 4

"Then the following group of statements ought to be true. Let's test them."

		If the number thought of is 9	If the number thought of is 4
E	n · n = (n + 1) · (n − 1) + 1		1
F	n · n = (n + 2) · (n − 2) + 4	4	16 = 6 · 2 + 4
G	n · n = (n + 3) · (n − 3) + 9	81 = 12 · 6 + 9	16 = 7 · 1 + 9
H	n · n = (n + 4) · (n − 4) + 16	81 = 13 · 5 + 16	16 = 8 · 0 + 16

"The following statements must also be true," Anne asserted.

		If the number thought of is 7	If the number thought of is 10
I	(n + 1) · (n − 1) = n · n − 1	8 · 6 = 49 − 1	11 · 9 = 100 − 1
J	(n + 2) · (n − 2) = n · n − 4	9 · 5 = 49 − 4	12 · 8 = 100 − 4
K	(n + 3) · (n − 3) = n · n − 9	10 · 4 = 49 − 9	13 · 7 = 100 − 9
L	(n + 4) · (n − 4) = n · n − 16	11 · 3 = 49 − 16	14 · 6 = 100 − 16



(115)

"There is a pattern running through the examples on the last three pages," Anne explained. "It is something like a game that might be called 'Think-of-Any-Two-Numbers.' Here's the way it works."

(Your own examples)

Think-of-Any Two Numbers	Shorthand	Ben	Ruth	Ellen	Roy	A	B	C
Your bigger number	b	7	6	10	15	10	10	12
Your smaller number	s	3	4	7	10	2	3	6
The sum of your numbers	b + s	10	10	17	25	12	13	18
The difference of your numbers	b − s	4	2	3	5	8	7	6
These multiplied together	(b + s)·(b − s)	40	20	51	125	96	91	108
The bigger number thought of times itself	b · b	49	36	100	225	100	100	144
The smaller number thought of times itself	s · s	9	16	49	100	4	9	36
Subtract these	b · b − s · s	40	20	51	125	96	91	108

"For any pair of numbers you think of, (b + s) · (b — s) = b · b — s · s."

"Many pairs of factors," Anne explained, can be rewritten in the form of (b + s) (b — s) and in the form of b · b — s · s. Often, these forms point to important shortcuts. Sometimes they do not help at all."

12 · 8	= (10 + 2) · (10 − 2)	= 10 · 10 − 2 · 2	= 100 − 4	= 96
10 · 16	= (13 − 3) · (13 + 3)	= 13 · 13 − 3 · 3	= 169 − 9	= 160
21 · 19	= (20 + 1) · (20 − 1)	= 20 · 20 − 1 · 1	= 400 − 1	= 399
22 · 18	= (20 + 2) · (20 − 2)	= 20 · 20 − 2 · 2	= 400 − 4	= 396
35 · 25	= (30 + 5) · (30 − 5)	= 30 · 30 − 5 · 5	= 900 − 25	= 875
53 · 47	= (50 + 3) · (50 − 3)	= 50 · 50 − 3 · 3	= 2500 − 9	= 2491
56 · 44	= (50 + 6) · (50 − 6)	= 50 · 50 − 6 · 6	= 2500 − 36	= 2464
109 · 91	= (100 + 9) · (100 − 9)	= 100 · 100 − 9 · 9	= 10,000 − 81	= 9,919
79 · 81	= (80 − 1) · (80 + 1)	= 80 · 80 − 1 · 1	= 1600 − 1	= 1599

Numbers	1		4		9		16
Differences		3		5		7	

(A, B, C, D) If we subtract 1 from 5 × 5, we get 6 × 4.
(E, F, G, H) If we add 1 to 6 × 4, we get 5 × 5.
(I, J, K, L) 6 × 4 is 1 less than 5 × 5.

Obviously, we could go on like this for a long time, with *five more* and *five less,* then *six more* and *six less,* and so on. We would notice a pattern; we would find ourselves able to guess the next result.

Notes to accompany *Work Text* page 115

Rows I, J, K, L of page 114 begin like this:

$$(n+1) \cdot (n-1) = n \cdot n - 1$$
$$(n+2) \cdot (n-2) = n \cdot n - 4$$
$$(n+3) \cdot (n-3) = n \cdot n - 9$$
$$(n+4) \cdot (n-4) = n \cdot n - 16$$

Is there a pattern here? What should the next statement in this series be? It is easy to guess it will begin with (n+5) · (n−5) = n·n − ____. But what number should finish it?

A student may say, "Why, 1, 4, 9, 16 are the square numbers: 1 × 1; 2 × 2; 3 × 3; 4 × 4. So the next number must be 5 × 5, which is 25." Another student may notice the steps by which 1, 4, 9, 16 grow:

The next difference should be 9, and the next number should be 25, which is quite correct! To be sure that this student also sees that 25 = 5 × 5, we might ask, "Do you see anything about the factors of 1, 4, 9, 16 that suggests a pattern?" Or we might ask, "Have we ever before met any numbers that grew by the odd numbers 3, 5, 7, 9 . . .?" The student may remember (or be helped to remember) the first exercise on *Work Text* page 101. One way or another, the students ought to see the pattern:

$$(n+1) \cdot (n-1) = n \cdot n - 1 \cdot 1$$
$$(n+2) \cdot (n-2) = n \cdot n - 2 \cdot 2$$
$$(n+3) \cdot (n-3) = n \cdot n - 3 \cdot 3$$
$$(n+4) \cdot (n-4) = n \cdot n - 4 \cdot 4$$
$$(n+5) \cdot (n-5) = n \cdot n - 5 \cdot 5$$

Once students grasp the pattern, they will understand that:

n + any number you like
times
n − that number
=
n times n
minus
that number times that number

It would be helpful if we had some shorthand for *any number you like.* Why not *a* for *any number?* Then:

$n + a$
times
$n - a$
=
n times n
minus
a times a

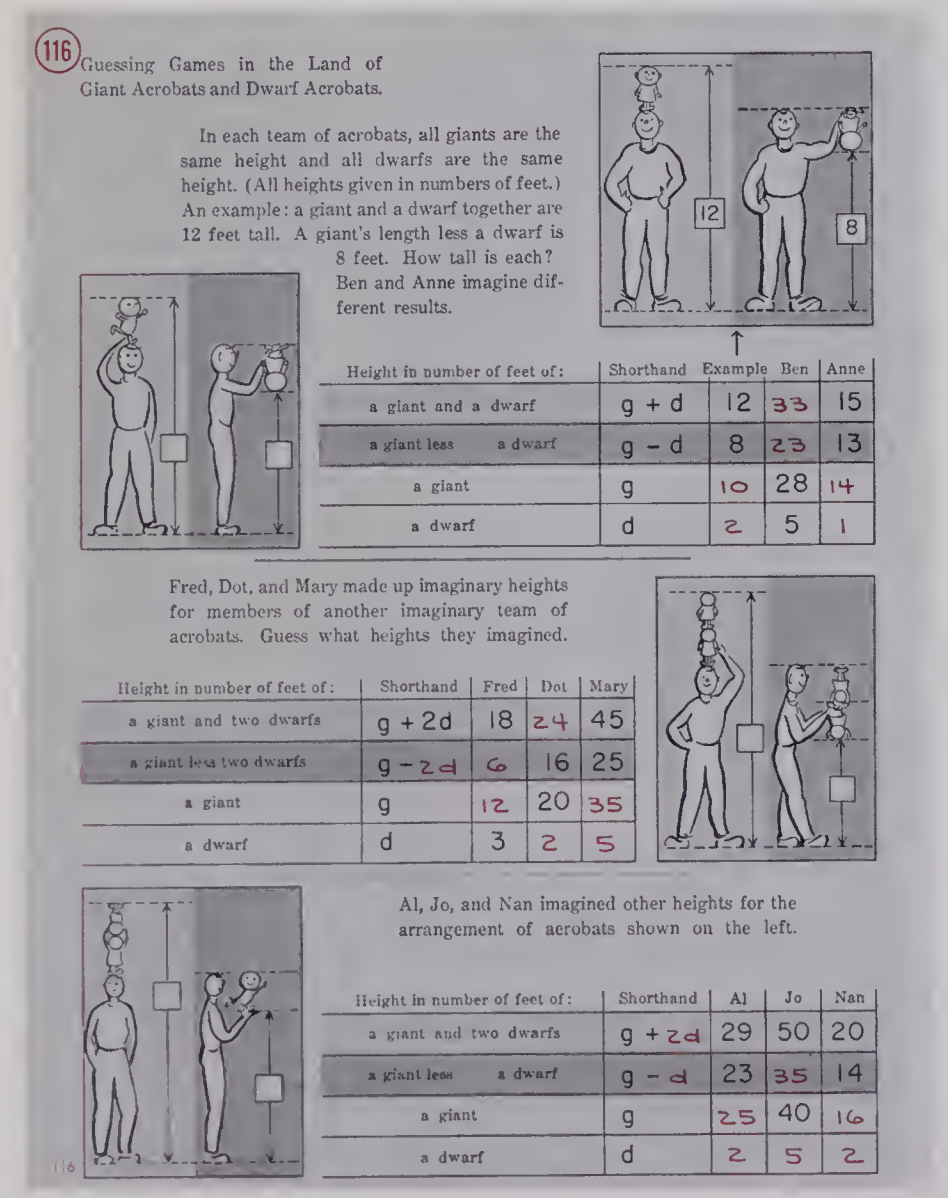
or, completely in shorthand:

$$(n+\text{a}) \cdot (n-\text{a}) = n \cdot n - a \cdot a$$

On page 115, we have actually not used n and a. We have put b for the bigger number and s for the smaller, so the result appears as

$$(b+s) \cdot (b-s) = b \cdot b - s \cdot s$$

The meaning is exactly the same. When we make an abbreviation for *the bigger number,* it does not matter whether we put b for bigger or n for number. The meaning is exactly the same. Whatever abbreviations your students suggest, accept them. The only test is, "Do you get true statements when you replace the abbreviations by suitable numbers?" For example, in the work above, the number that replaces a must be smaller than the number that replaces n. Otherwise, we shall be in trouble when we come to $(n-a)$.



116 Guessing Games in the Land of Giant Acrobats and Dwarf Acrobats.

In each team of acrobats, all giants are the same height and all dwarfs are the same height. (All heights given in numbers of feet.) An example: a giant and a dwarf together are 12 feet tall. A giant's length less a dwarf is 8 feet. How tall is each? Ben and Anne imagine different results.

12 8

Height in number of feet of:	Shorthand	Example	Ben	Anne
a giant and a dwarf	g + d	12	33	15
a giant less a dwarf	g - d	8	23	13
a giant	g	10	28	14
a dwarf	d	2	5	1

Fred, Dot, and Mary made up imaginary heights for members of another imaginary team of acrobats. Guess what heights they imagined.

Height in number of feet of:	Shorthand	Fred	Dot	Mary
a giant and two dwarfs	g + 2d	18	24	45
a giant less two dwarfs	g - 2d	6	16	25
a giant	g	12	20	35
a dwarf	d	3	2	5

Al, Jo, and Nan imagined other heights for the arrangement of acrobats shown on the left.

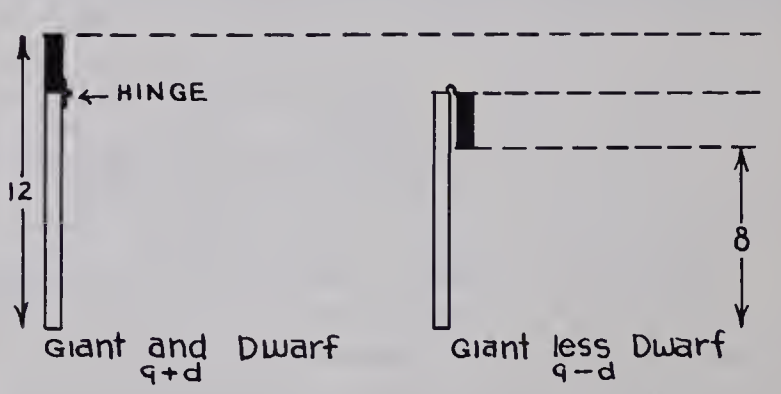
Height in number of feet of:	Shorthand	Al	Jo	Nan
a giant and two dwarfs	g + 2d	29	50	20
a giant less a dwarf	g - d	23	35	14
a giant	g	25	40	16
a dwarf	d	2	5	2

116

Notes to accompany *Work Text* page 116

The title of the game on page 116 is important: "A Guessing Game." Obviously, the problems here resemble those in a formal algebra course, under the heading *simultaneous equations.* But the purpose of *Work Text* pages 116-119 is *not* to teach students any formal methods of solving equations. It is a more modest aim. We would like the students to get used to expressions such as $g+d$ and $g-d$, so that they are able to use them without fear. We would like students to appreciate the meaning of $g+d$ and $g-d$; and the pictures of giants and dwarfs help to convey that meaning. The students learn to test their solutions. It is most important for a student of mathematics to check his solutions. Students who do not have this habit often talk nonsense.

Guessing can be aided by vision. The first problem can be illustrated by two objects hinged together.

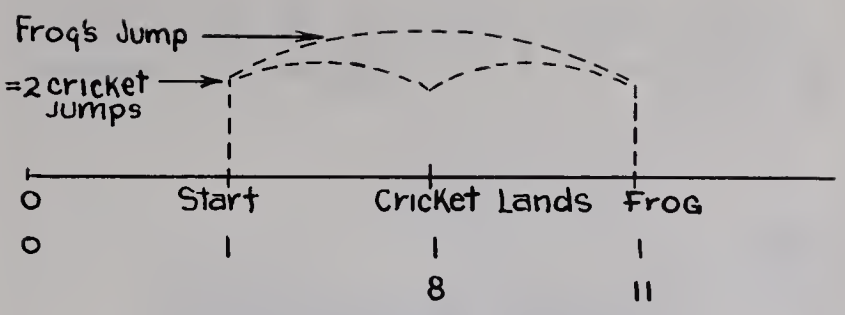


It is surely evident (1) that two dwarfs would fit into the space between heights 8 feet and 12 feet; and (2) that the giant's height is midway between 8 feet and 12 feet. The above model can be improvised by folding a long strip of paper, or by using pencils of unequal lengths to represent the giant and the dwarf.

In the same way, in Mary's problem, there is seen to be room for 4 dwarfs between heights 25 feet and 45 feet. In Al's problem, there is room for 3 dwarfs between the heights of 23 feet and 29 feet. A similar remark applies to Nan's problem.

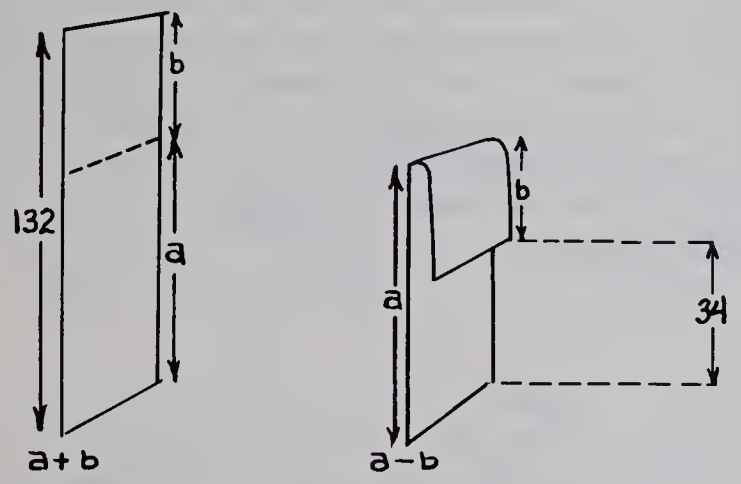
Notes to accompany *Work Text* page 117

The work on this page closely resembles that on page 116. The students should have little difficulty with the Frog and Cricket problem if they make a drawing to record the information:



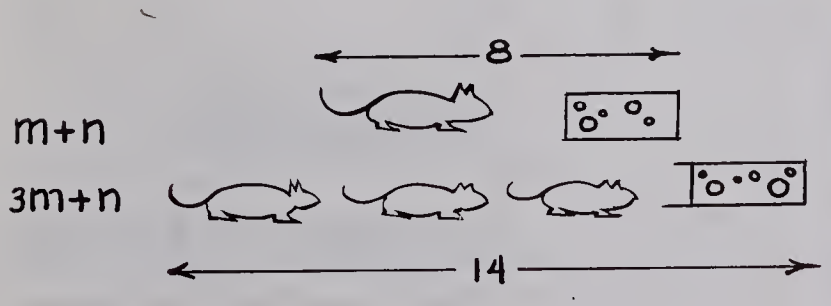
Ben tells us the Cricket lands at 8 and the Frog at 11. The distance between 8 and 11 is seen to be one cricket's jump.

In working out most of the problems about Andy's bag and Bill's bag, students should be asked to guess the answers without any help at all. Where large numbers are involved, this may not be possible. In such cases, if the students have not already hit on some similar idea, point out to them that $a+b$ and $a-b$ can be pictured with a strip of paper in the same way that $g+d$ and $g-d$ were pictured. For example, the last Andy and Bill problem may be illustrated like this:



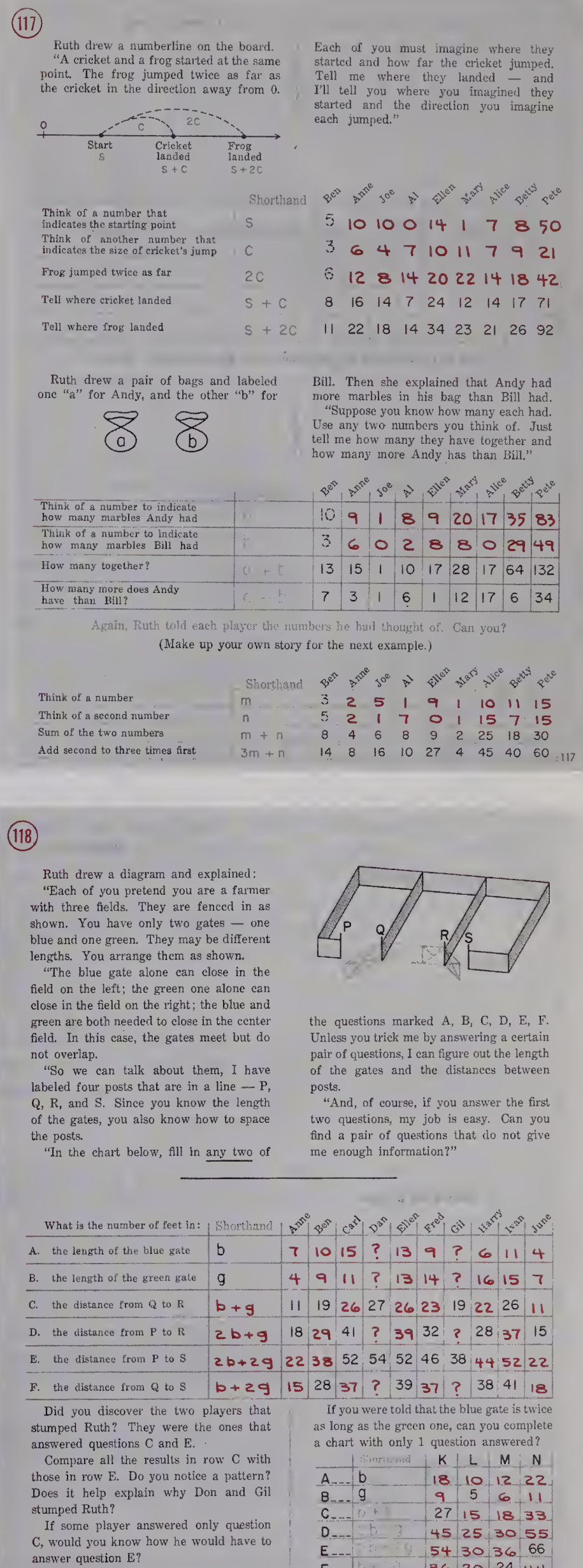
The picture suggests that there is room for $2b$ between 34 and 132. Since 132 − 34 = 98, it follows that b must be 49. Therefore, Bill had 49 marbles and Andy had 83 marbles (49+34).

The final problem can be pictured similarly. If m inches is the length of a mouse and n inches is the length of a piece of cheese, we can draw:



Notes to accompany *Work Text* page 118

The following pictures illustrate the various problems on *Work Text* page 118:



(117)

Ruth drew a numberline on the board. "A cricket and a frog started at the same point. The frog jumped twice as far as the cricket in the direction away from 0.

Each of you must imagine where they started and how far the cricket jumped. Tell me where they landed — and I'll tell you where you imagined they started and the direction you imagine each jumped."

	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number that indicates the starting point	S	5	10	10	0	14	1	7	8	50
Think of another number that indicates the size of cricket's jump	C	3	6	4	7	10	11	7	9	21
Frog jumped twice as far	2C	6	12	8	14	20	22	14	18	42
Tell where cricket landed	S + C	8	16	14	7	24	12	14	17	71
Tell where frog landed	S + 2C	11	22	18	14	34	23	21	26	92

Ruth drew a pair of bags and labeled one "a" for Andy, and the other "b" for Bill. Then she explained that Andy had more marbles in his bag than Bill had.

"Suppose you know how many each had. Use any two numbers you think of. Just tell me how many they have together and how many more Andy has than Bill."

		Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number to indicate how many marbles Andy had	a	10	9	1	8	9	20	17	35	83
Think of a number to indicate how many marbles Bill had	b	3	6	0	2	8	8	0	29	49
How many together?	a + b	13	15	1	10	17	28	17	64	132
How many more does Andy have than Bill?	a − b	7	3	1	6	1	12	17	6	34

Again, Ruth told each player the numbers he had thought of. Can you?

(Make up your own story for the next example.)

	Shorthand	Ben	Anne	Joe	Al	Ellen	Mary	Alice	Betty	Pete
Think of a number	m	3	2	5	1	9	1	10	11	15
Think of a second number	n	5	2	1	7	0	1	15	7	15
Sum of the two numbers	m + n	8	4	6	8	9	2	25	18	30
Add second to three times first	3m + n	14	8	16	10	27	4	45	40	60

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(118)

Ruth drew a diagram and explained:

"Each of you pretend you are a farmer with three fields. They are fenced in as shown. You have only two gates — one blue and one green. They may be different lengths. You arrange them as shown.

"The blue gate alone can close in the field on the left; the green one alone can close in the field on the right; the blue and green are both needed to close in the center field. In this case, the gates meet but do not overlap.

"So we can talk about them, I have labeled four posts that are in a line — P, Q, R, and S. Since you know the length of the gates, you also know how to space the posts.

"In the chart below, fill in any two of the questions marked A, B, C, D, E, F. Unless you trick me by answering a certain pair of questions, I can figure out the length of the gates and the distances between posts.

"And, of course, if you answer the first two questions, my job is easy. Can you find a pair of questions that do not give me enough information?"

What is the number of feet in:	Shorthand	Anne	Ben	Carl	Dan	Ellen	Fred	Gil	Harry	Ivan	June
A. the length of the blue gate	b	7	10	15	?	13	9	?	6	11	4
B. the length of the green gate	g	4	9	11	?	13	14	?	16	15	7
C. the distance from Q to R	b + g	11	19	26	27	26	23	19	22	26	11
D. the distance from P to R	2b + g	18	29	41	?	39	32	?	28	37	15
E. the distance from P to S	2b + 2g	22	38	52	54	52	46	38	44	52	22
F. the distance from Q to S	b + 2g	15	28	37	?	39	37	?	38	41	18

Did you discover the two players that stumped Ruth? They were the ones that answered questions C and E.

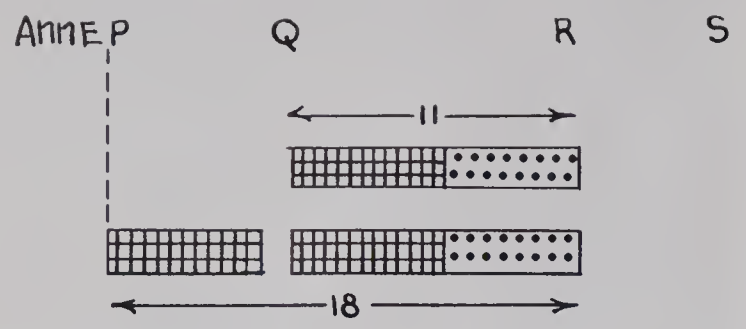
Compare all the results in row C with those in row E. Do you notice a pattern? Does it help explain why Don and Gil stumped Ruth?

If some player answered only question C, would you know how he would have to answer question E?

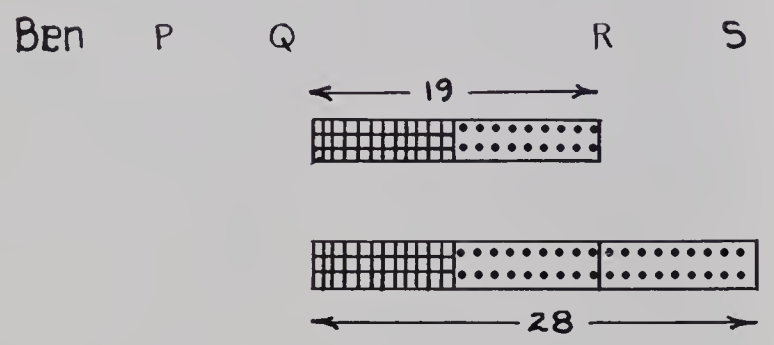
If you were told that the blue gate is twice as long as the green one, can you complete a chart with only 1 question answered?

	Shorthand	K	L	M	N
A	b	18	10	12	22
B	g	9	5	6	11
C	b + g	27	15	18	33
D	b + 3g	45	25	30	55
E	3b + g	54	30	36	66
F	b + 4g	36	20	24	44

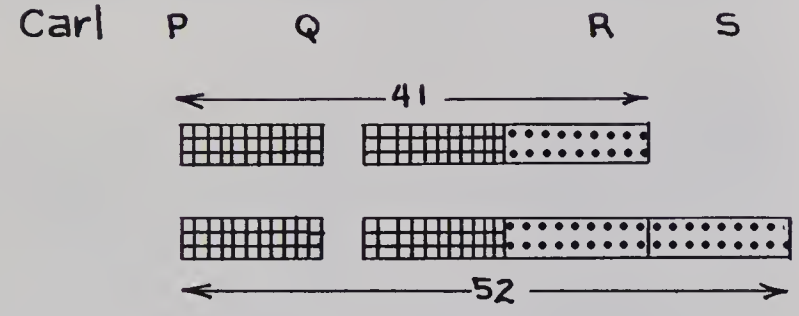
118



From this, it is fairly clear that the blue gate, from P to Q, is 18-11 = 7 feet.

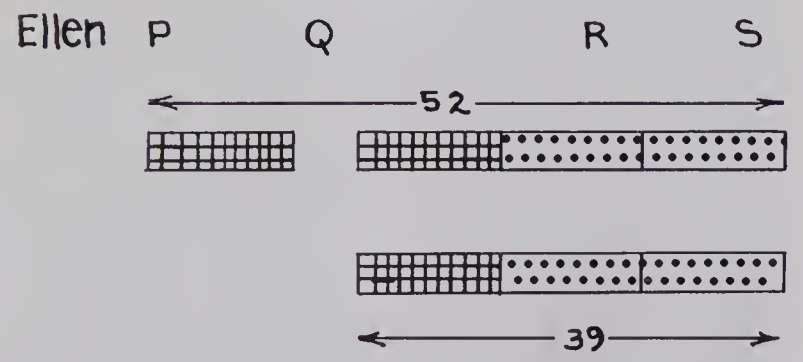


The green gate, from R to S, must be 28-19 = 9 feet.

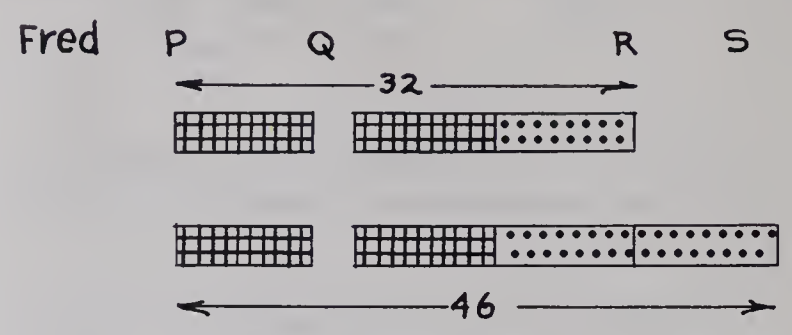


Green, from R to S, must be 52-41 = 11 feet. P to R, 41 feet, therefore contains two blue spaces and 11 feet. The two blue spaces occupy 30 feet. So, b = 15.

Dan – Any two numbers that add up to 27 will do. Ruth was unable to say what numbers Dan had chosen.



Blue is 52-39 = 13 feet long. So, two greens and 13 add up to 39. Two greens, therefore, make 26. So, $g = 13$.

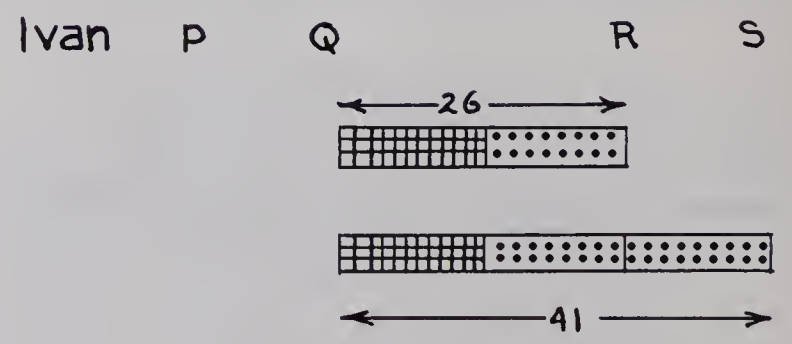


Green must be 46 – 32 = 14. As two blues and a green fill the 32 feet from P to R, and green is 14, the two blues must make 18. So, $b = 9$.

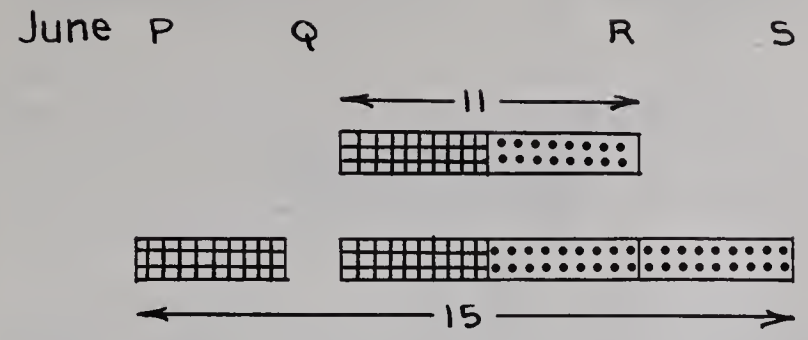
Gil – Any two numbers that add up to 19 will do for b and g. This is the second problem that beat Ruth.

Harry – The pictures do not help us much with this one, and the students may not see any simple way to reason it out. Guesswork is probably the quickest way to arrive at an answer. We can find many ways of making 2b + 8 = 28. From all of these, we want one that makes b + 2g = 38. The work might go like this:

Guesses	Make $2b+g = 28$	*Find b+2g*	*Result*
1) $b = 10$	$g = 8$	26	Too small.
2) $b = 11$	$g = 6$	23	Even worse! Go in opposite
3) $b = 9$	$g = 10$	29	direction. Better than the
4) $b = 8$	$g = 12$	32	first two guesses!
5) $b = 7$	$g = 14$	35	Still better.
6) $b = 6$	$g = 16$	*38*	Very close now. This is it.



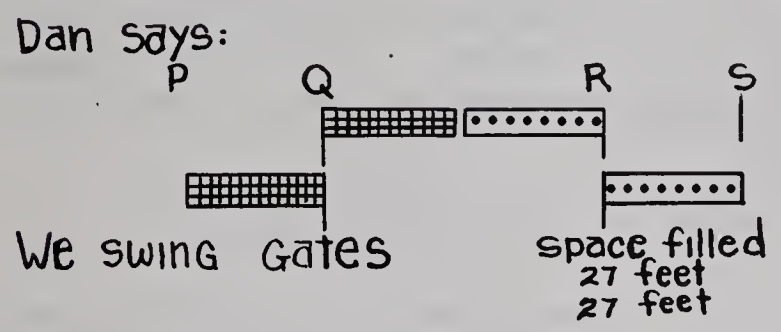
The green gate, from R to S, must be 41 – 26 = 15 feet. The blue gate must be 11 feet.



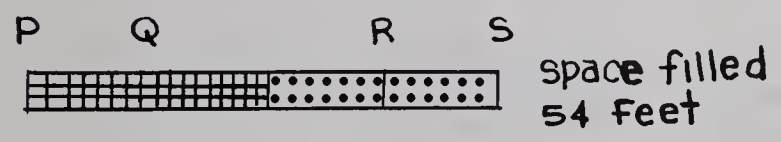
The blue gate, from P to Q, must be 15 - 11 = 4 feet. So, the green gate must be 7 feet.

Why Ruth cannot answer Dan and Gil

The numbers in row E are exactly twice the numbers in row C. This is the pattern. Why does it happen? Consider Dan's first statement. A blue gate and a green gate occupy 27 feet. If we swing them both around, they will still occupy 27 feet.



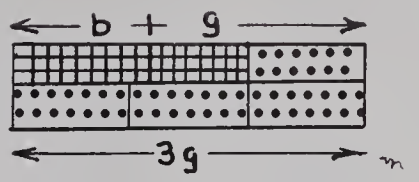
Put these two sets of gates together. We get:



So, if *one* blue gate and *one* green gate fill 27 feet, automatically *two* blue gates and *two* green gates will fill 54 feet. *Dan's second statement tells us nothing new.* You can choose any numbers for b and g that make b + g = 27; automatically 2b + 2g will be 54. (The students should test this statement by choosing particular numbers.)

Last problem

A blue gate is twice as long as a green gate. So a blue gate and a green gate have the same total length as 3 green gates.



In shorthand, $b + g = 3g$. In *column K,* we are told $b + g = 27$. So, $3g = 27$. Therefore, $g = 9$. The blue gate is twice as long as the green gate. So, $b = 18$. Now we know the sizes of both gates, and the rest is easy.

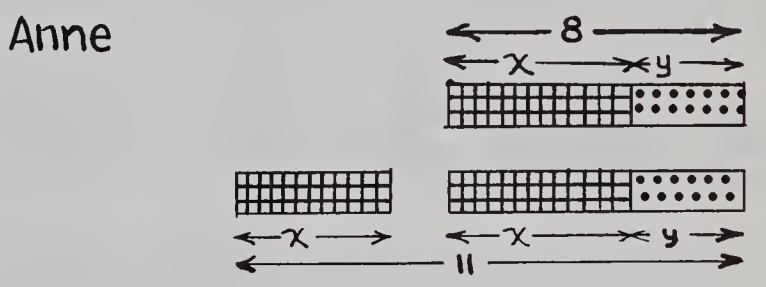
Column L is also easy. We are told that the green gate is 5 feet. The blue gate is twice as long, so it must be 10 feet.

In *column M,* we are told the length formed by one blue and two green gates. This is the same length as four green gates (students should draw a picture if they doubt this). So, the green gate must be 6 feet. Complete problem as in column L.

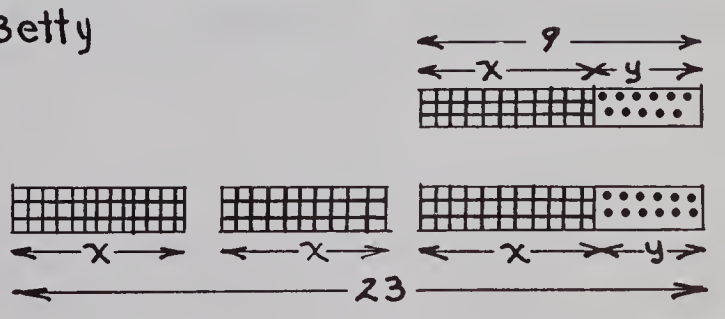
Finally, in *column N,* two blue and two green gates have the same length as 6 green gates. So, the green gate must be 11 feet. Complete as in columns L and M.

Notes to accompany *Work Text* page 119

All of the first set of problems become easy once pictures are drawn. Fred's problem requires one more step than the others. It is quite acceptable if students solve these problems by their own methods, or if they guess the answers. We show x by a blue block, and y by a green.



Adding x changes 8 to 11, so x must be 3. Then y must be 5, since $3 + 5 = 8$. Since $(2\times3) + 5 = 11$, both statements are checked. No other solution is possible: the argument shows that x *must* be 3 and y *must* be 5.



119 "Here's another game," Ruth began. "Each of you think of any two numbers. Then tell me what you get for any two of the six expressions given below. I will tell you what you would get for the other expressions. I will also tell you the two numbers you thought of."

Player	one number	other number						
Anne	3	5	8	11	13	14	19	29
Betty	7	2	9	16	11	23	25	29
Carl	5	5	10	15	15	20	25	35
Dan	10	9	19	29	28	39	48	66
Ellen	15	1	16	31	17	46	47	49
Fred	4	11	15	19	26	23	34	56
Gil	2	4	6	8	10	10	14	22

"Think of two other numbers — one larger than the other. The rest of the rules are the same. I will change the expressions. Tell me what you get for any two."

	bigger number b	smaller number s	b+s	b−s	b+2s	2b−s	2b+s	3b−s
Anne	10	5	15	5	20	15	25	25
Betty	9	8	17	1	25	10	26	19
Carl	4	1	5	3	6	7	9	11
Dan	12	2	14	10	16	22	26	34
Ellen	15	5	20	10	25	25	35	40
Fred	6	0	6	6	6	12	12	18
Gil	7	4	11	3	15	10	18	17

"Again think of two numbers. What do you get as results for two of the five statements? I will tell you the numbers you thought of."

		Anne	Betty	Carl	Dan	Ellen	Fred	Gil
Think of a number	p	6	7	10	12	20	11	7
Think of a smaller number	q	2	5	3	7	9	5	0
The sum of your numbers	p+q	8	12	13	19	29	16	7
The smaller number plus twice the larger number	q+2p	14	19	23	31	49	27	14
The difference of your numbers	p−q	4	2	7	5	11	6	7
From five times the larger number subtract the smaller number	5p−q	28	30	47	53	91	50	35
Three times the larger number plus twice the smaller number	3p+2q	22	31	36	50	78	43	21

119

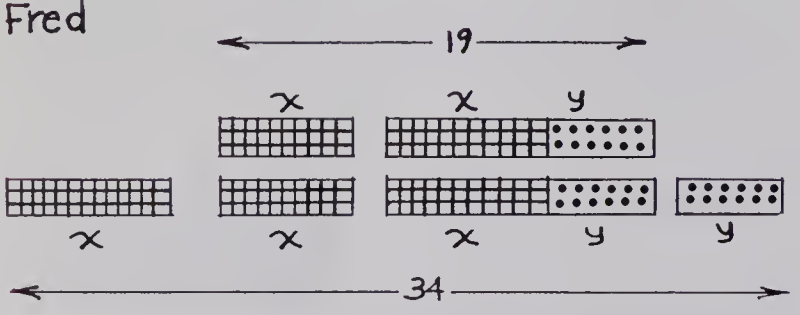
Two extra blue blocks increase the length from 9 to 23. They bring in 14 units. Each must be 7. Therefore, $x = 7$, and y must be 2. This checks for both 9 and 23.

Carl — a similar picture shows that two extra blue blocks increase the length from 15 to 25. So each blue block is 5. Thus, $x = 5$ and $y = 5$.

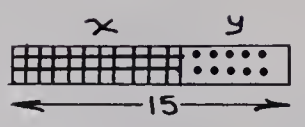
Dan — 19 is increased to 28 when a single green block is added. So $y = 28 - 19 = 9$: x must be 10. Check.

Ellen — a single blue box is added and 31 is increased to 46. So. $x = 46 - 31 = 15$; y must be 1.

Fred



In considering Fred's problem, students often produce the following argument: The length increased from 19 to 34 when we added one x and one y block. Since $34 - 19 = 15$, we must have $x + y = 15$.



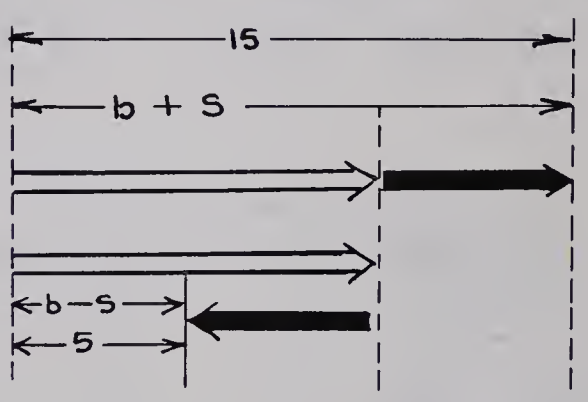
Compare this with the top picture for 19. An additional x makes the difference between 15 and 19. So $x = 4$. Then y must be 11.

Gil — A picture shows that a length of 10 units increases to 22 when *three* green blocks are added. Since $22 - 10 = 12$, each green block must be 4. So $y = 4$. From the picture for $3x + y = 10$, we see that 3 blue blocks and 1 green (now known to be 4) makes up 10. So 3 blue make 6. Thus, $x = 2$.

The b and s problems

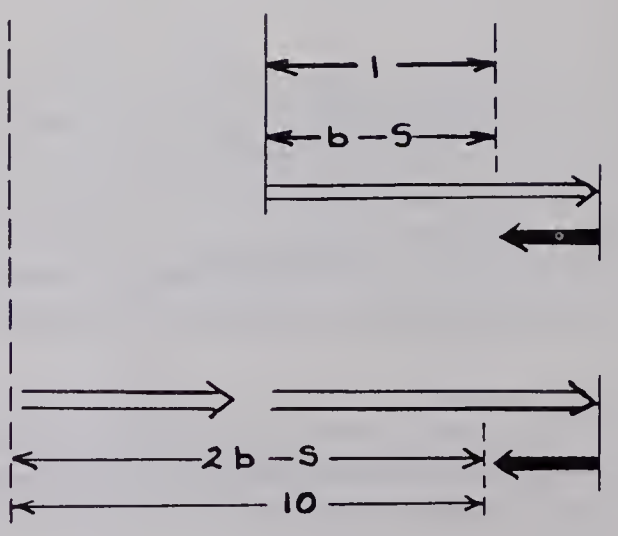
These problems can be pictured, if the students like, by giants (b for big) and dwarfs (s for small). See the illustrations of addition and subtraction on *Work Text* page 116. More compact, but less exciting, illustrations are shown here. We represent b by an arrow ⟹ and s by an arrow ➡. When s is subtracted, this arrow will point to the left.

Anne

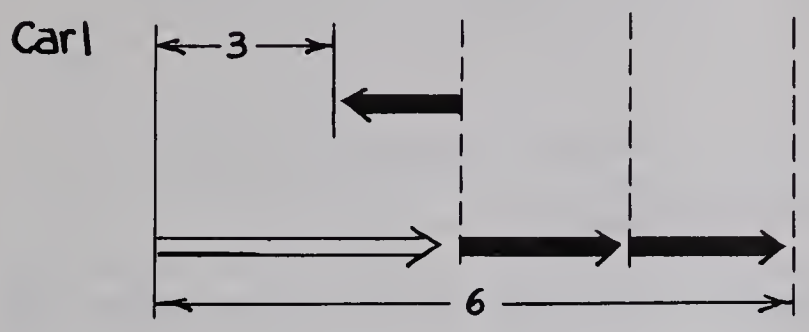


It is seen that two black arrows fit into the space between 5 and 15. So s must be 5. This means $b = 10$.

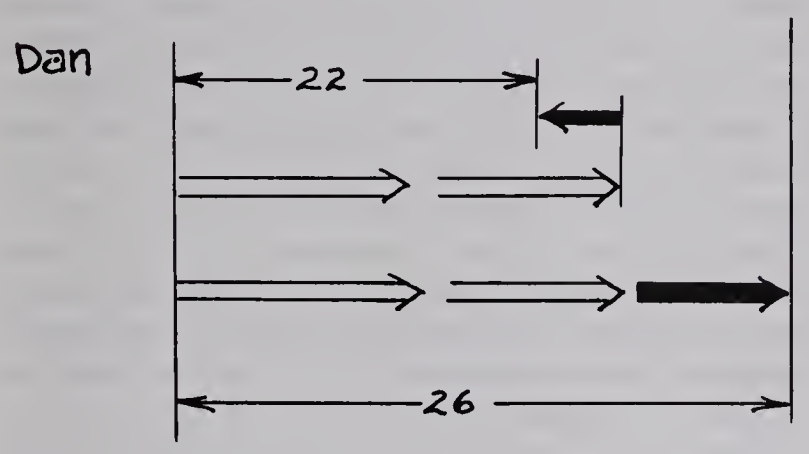
Betty



Adding the extra b changes 1 to 10. So b must be 9. Then $s = 8$. This picture is a little complicated. Students may prefer to guess. Since $b - s = 1$, we know that b is one more than s. So students could try pairs of numbers for b and s, such as 5 and 4, or 13 and 12, and keep on until they found a pair that made 2b - s equal to 10.



Three black arrows fit into the space between 3 and 6. Each must be 1. So $s = 1$ and $b = 4$.



We see $2s = 26 - 22 = 4$. So $s = 2$. If 2b + 2 = 26, then $2b$ must be 24, so $b = 12$.

Ellen—This problem resembles Carl's. The picture will show that $3b - s$ exceeds $2b - s$ by b. So this extra b makes the difference between 25 and 40. So, $b = 40 - 25 = 15$. Then $2b$ is 30. Since $2b - s = 25$, it follows that $s = 5$, for 5 is what we have to subtract from 30 to give 25.

Fred — We might think of numbers that add up to 6. If we guess $b = 4$, $s = 2$, we find $3b - s = 10$; too small, we wanted 18. If we tried $b = 3$, $s = 3$, we would get $3b - s = 6$, even worse. So we try going in the opposite direction. If $b = 5$, $s = 1$, we find $3b - s = 14$. There seems no way out but to try $b = 6$, $s = 0$, and this does the trick.

A much more sophisticated approach to the same problem would be to reason: if one b and one s add up to 6, two b and two s will add up to twice as much, or 12. Three b and three s will be three times as much, or 18. So $3b + 3s = 18$. But we are told (in the last column) that $3b - s = 18$. Apparently it makes no difference whether we add three s or subtract one s! There is only one number for which addition and subtraction have the same effect—zero. So we try $s = 0$, and it works.

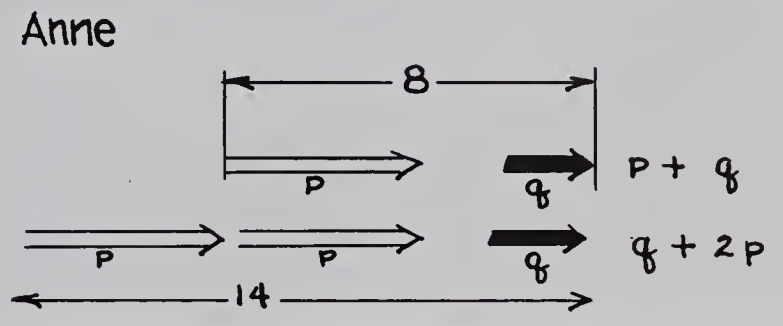
Do not press this last argument on students. If they are not thoroughly familiar with the "Do-It-To-Both" Rule, they will not follow the reasoning. No harm is done by omitting this argument, if students are not ready for it.

Gil — The easiest and quickest way to complete this row is to guess. $b - s = 3$, so b must be 3 more than s. Have students try different values in order to find $2b + s$ for each.

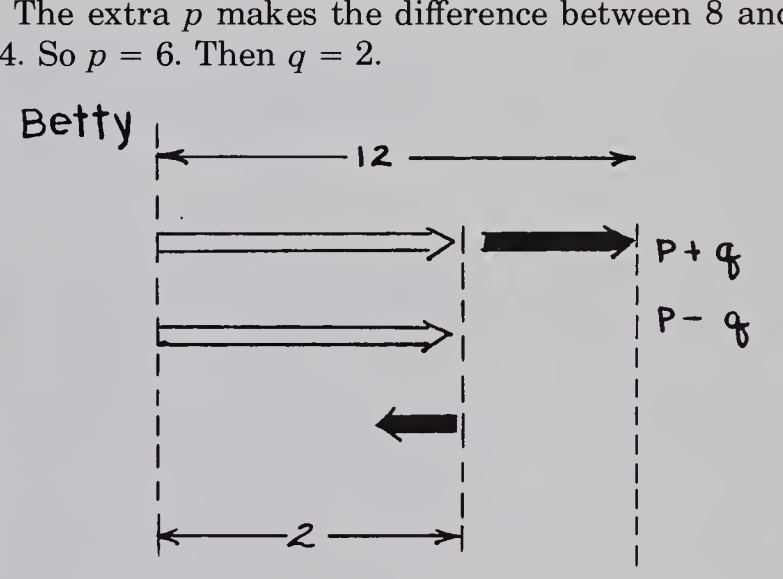
b	3	4	5	6	7
s	0	1	2	3	4
2b+s	6	9	12	15	18

So $b = 7$, $s = 6$ is the one we want.

The p and q problems



The extra p makes the difference between 8 and 14. So $p = 6$. Then $q = 2$.

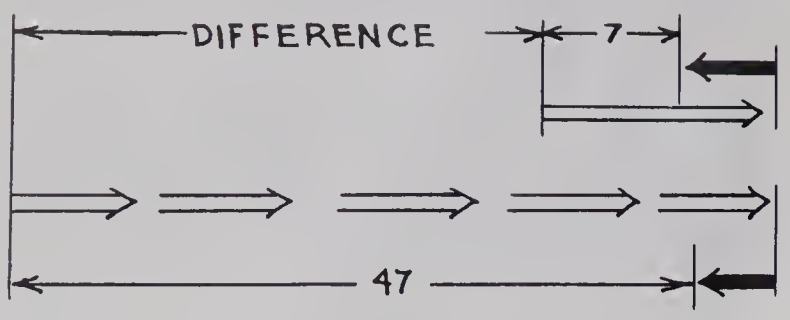


Two black arrows fit into the space between 2 and 12. So $2q = 12 - 2 = 10$. Hence $q = 5$ and $p = 7$.

Carl — Two methods are possible. (1) Guess: $p - q = 7$, so we need only consider pairs of numbers where p is larger than q by 7. Try different guesses until we find one that makes $5p - q = 47$.

p	7	8	9	10
q	0	1	2	3
5p-q	35	39	43	47

Method (2). A picture will show that $5p - q$ exceeds $p - q$ by $4p$.



So $4p$ makes the difference $47 - 7 = 40$. So $p = 10$. Since $10 - 3 = 7$, q must be 3.

Dan — This problem is exactly like Betty's.

Ellen — Same method as Anne's.

Fred — Either method used for Carl's problem will work here.

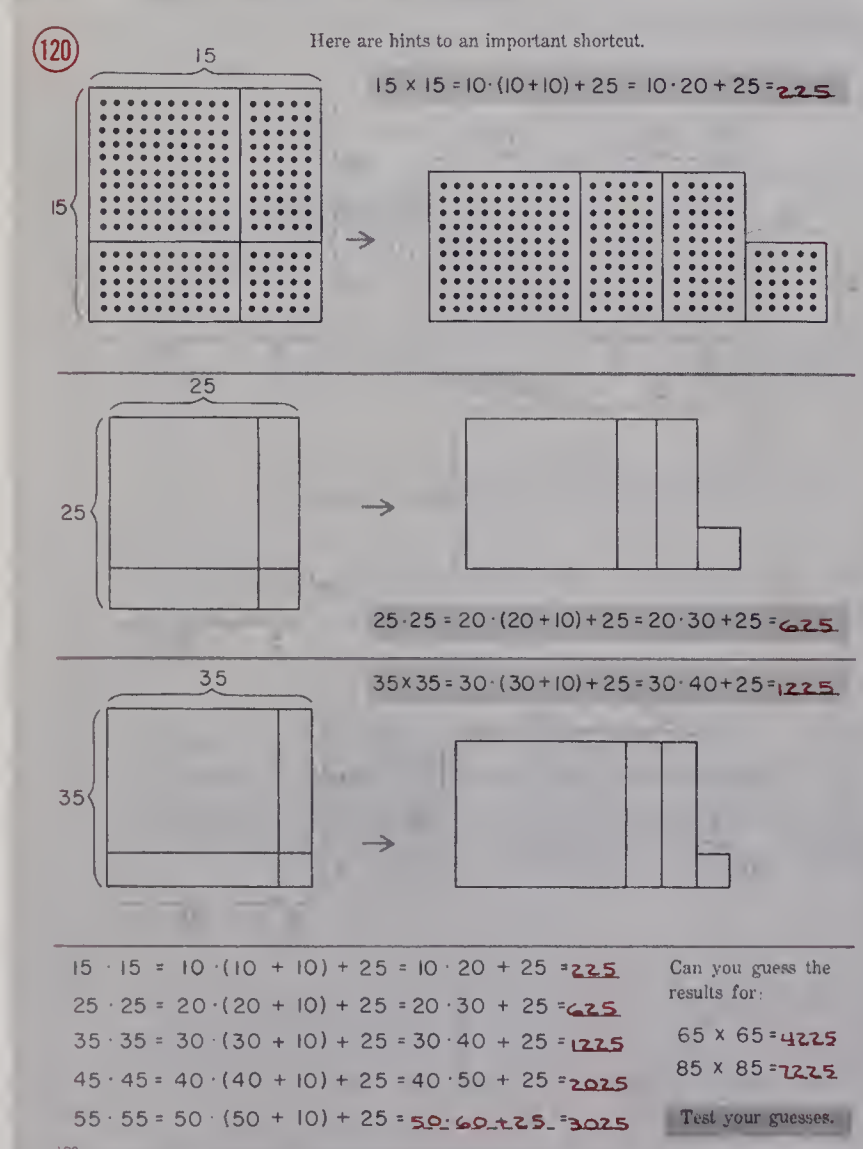
Gil — Guesswork is in order. $5p - q = 35$ suggests that $5p$ is more than 35, so p looks as if it would be more than 7. We may begin by trying 8, 9 . . . for p. To make $5p - q = 35$, we could have $p = 8$, $q = 5$, or $p = 9$, $q = 10$. We try numbers such as these and see if any of them make $3p + 2q$ equal to 21. But:

p	8	9	10
q	5	10	15
3p+2q	34	47	60

We are getting further and further away from 21, the result we want. We try the opposite direction:

p	7	8	9	10
q	0	5	10	15
3p+2q	21	34	47	60

We are home. $p = 7$, $q = 0$.



Notes to accompany *Work Text* page 120

There is a well-known rule for multiplying a number by itself if its final digit is 5. Suppose, for example, we want to find 75×75. The rule works like this: 75 lies between 70 and 80. Multiply these two numbers together. $70 \times 80 = 5600$. To this product add 25. This gives 5625, the correct answer. Page 120 tries to show why this rule works. We start with the simplest example, 15×15. First we draw 15 rows of 15 dots to represent 15×15. Then we cut this up, and put the pieces together so that we have 10 rows of 20 dots, *plus* a square containing 25 dots. This shows that 15×15 is the result of adding 25 to 10×20.

We then consider the next example, 25×25. We do not draw all the dots, but they should be imagined. We show how to cut the picture up and put it together again so that it shows 20 rows of 30 dots, together with a square of 25 dots. This work illustrates the result $25 \times 25 = 20 \times 30 + 25$.

The third picture illustrates $35 \times 35 = 30 \times 40 + 25$.

Notes to accompany *Work Text* page 121

This page begins with a series of examples that involve the rule presented on *Work Text* page 120.

The following ten lines require the student to remember the ideas and the results of the investigation on *Work Text* pages 112-115. You may aid their memory by asking questions such as, "Is 6×4 more or less than 5×5? How much? What is the difference between 14×12 and 13×13? Between 218×216 and 217×217?" This last question is *mental arithmetic;* students do not need to know what 217×217 is. They need only remember that the product of the end numbers: 218×216 is 1 less than the square of the middle number: 217×217. In shorthand, $(n+1) \cdot (n-1) = n \cdot n - 1$.

Can the students remember what happens with $n + 2$ and $n - 2$? What is the difference between 7×3 and 5×5? Between 12×8 and 10×10? This review should remind the students that $(n+2) \cdot (n-2)$ is always the same as $n \cdot n - 4$. Guide students along the line of thought followed on pages 112-115, so that they think through again to the conclusion: $(n+a) \cdot (n-a) = n \cdot n - a \cdot a$, or $(b+s) \cdot (b-s) = b \cdot b - s \cdot s$. This should prepare students for the exercises.

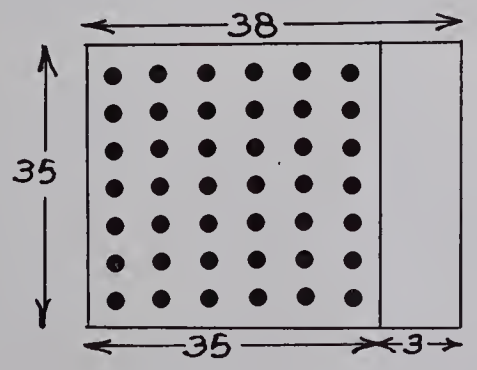
In the first exercise, we want to multiply 23 and 17. Now 20 comes midway between these numbers. In fact, $23 = 20 + 3$ and $17 = 20 - 3$. We know that multiplying these numbers gives us 9 less than 20×20. It is easier to find $400 - 9$ than to multiply out 23×17 by the usual method.

In the next exercise, we want 27×23. Midway between these numbers is 25. Since $27 = 25 + 2$ and $23 = 25 - 2$, we know that 27×23 will be 4 less than 25×25. We have 25×25 written at the top of the page. It is 625. So 27×23 must be $625 - 4 = 621$.

All the other exercises in this section uses a similar method. For b in the formula, we always choose *the number midway between the numbers we want to multiply.* For example, in the last question, we have 49×61. Midway between 49 and 61 is 55. Then $49 = 55 - 6$ and $61 = 55 + 6$. So we know that 49×61 will be 36 less than 55×55.

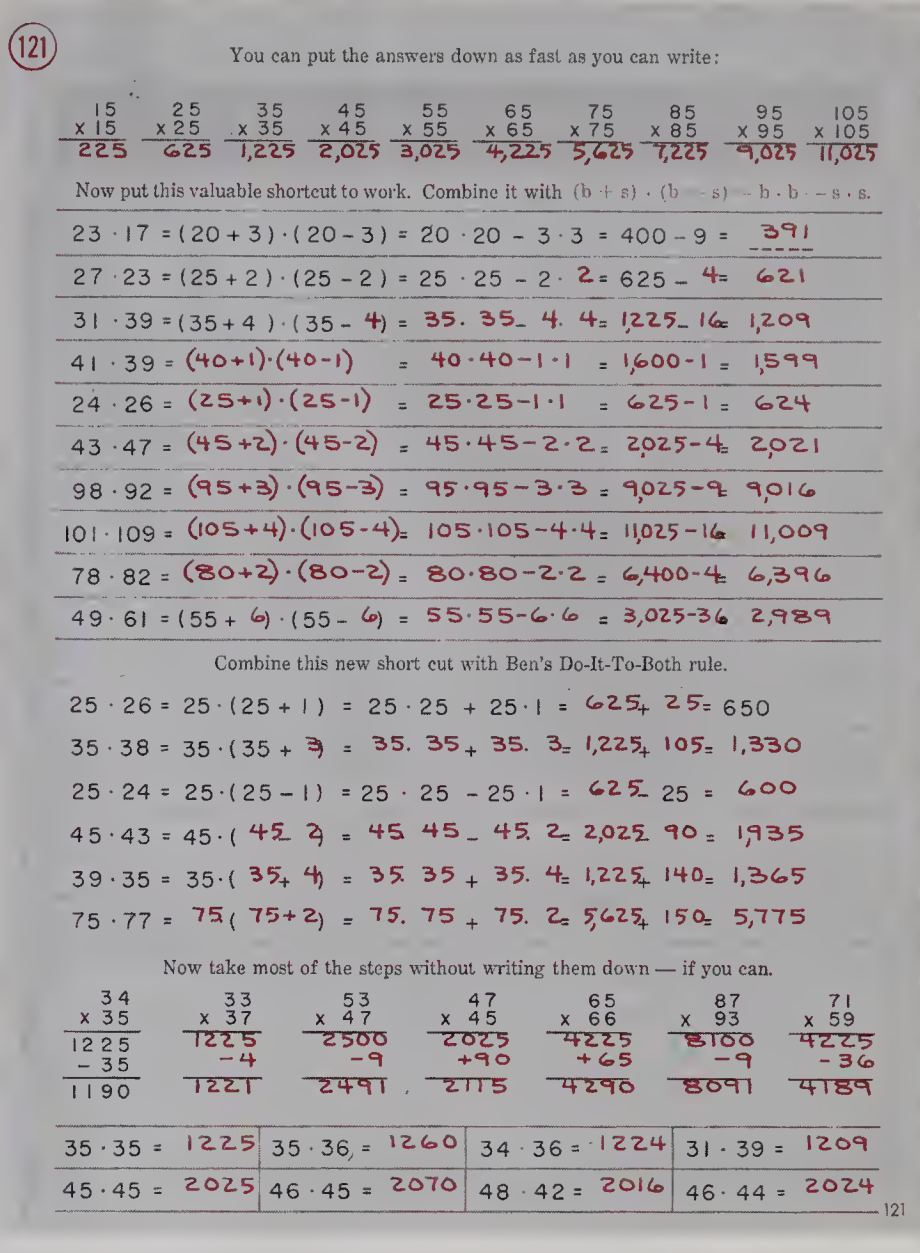
The next 6 lines combine the results at the top of the page with Ben's "Do-It-to-Both" rule. (See page 20.)

For instance, when we want 35×38 we see that this is not far from 35×35. In fact, 35 rows of 38 dots can be broken into 35 rows of 35 dots and 35 rows of 3 dots. The results at the top of the page tell us $35 \times 35 = 1225$. So there must be 1225 dots in the square shown here, and 105 dots in the tall rectangle at the right. The total number of dots in 35×38 must be the sum of these numbers— $1225 + 105$. All six exercises are similar.



With regard to the mental work at the bottom of the page: The rule of page 120 is used to find $35 \cdot 35$ and $45 \cdot 45$. The formula $(b+s) \cdot (b-s) = b \cdot b - s \cdot s$ is used to find 33×37; 53×47; 87×93; 71×59; $34 \cdot 36$; $48 \cdot 42$; $31 \cdot 39$ and $46 \cdot 44$.

Ben's "Do-It-to-Both" rule is used for 34×35; 47×45; 65×66; $35 \cdot 36$ and $46 \cdot 45$.


121

You can put the answers down as fast as you can write:

15 x 15	25 x 25	35 x 35	45 x 45	55 x 55	65 x 65	75 x 75	85 x 85	95 x 95	105 x 105
225	625	1,225	2,025	3,025	4,225	5,625	7,225	9,025	11,025

Now put this valuable shortcut to work. Combine it with $(b+s) \cdot (b-s) = b \cdot b - s \cdot s$.

$23 \cdot 17 = (20+3) \cdot (20-3) = 20 \cdot 20 - 3 \cdot 3 = 400 - 9 =$ 391

$27 \cdot 23 = (25+2) \cdot (25-2) = 25 \cdot 25 - 2 \cdot 2 = 625 - 4 =$ 621

$31 \cdot 39 = (35+4) \cdot (35-4) = 35 \cdot 35 - 4 \cdot 4 = 1225 - 16 =$ 1,209

$41 \cdot 39 = (40+1) \cdot (40-1) = 40 \cdot 40 - 1 \cdot 1 = 1600 - 1 =$ 1,599

$24 \cdot 26 = (25+1) \cdot (25-1) = 25 \cdot 25 - 1 \cdot 1 = 625 - 1 =$ 624

$43 \cdot 47 = (45+2) \cdot (45-2) = 45 \cdot 45 - 2 \cdot 2 = 2025 - 4 =$ 2,021

$98 \cdot 92 = (95+3) \cdot (95-3) = 95 \cdot 95 - 3 \cdot 3 = 9025 - 9 =$ 9,016

$101 \cdot 109 = (105+4) \cdot (105-4) = 105 \cdot 105 - 4 \cdot 4 = 11025 - 16 =$ 11,009

$78 \cdot 82 = (80+2) \cdot (80-2) = 80 \cdot 80 - 2 \cdot 2 = 6400 - 4 =$ 6,396

$49 \cdot 61 = (55+6) \cdot (55-6) = 55 \cdot 55 - 6 \cdot 6 = 3025 - 36 =$ 2,989

Combine this new short cut with Ben's Do-It-To-Both rule.

$25 \cdot 26 = 25 \cdot (25+1) = 25 \cdot 25 + 25 \cdot 1 = 625 + 25 =$ 650

$35 \cdot 38 = 35 \cdot (35+3) = 35 \cdot 35 + 35 \cdot 3 = 1225 + 105 =$ 1,330

$25 \cdot 24 = 25 \cdot (25-1) = 25 \cdot 25 - 25 \cdot 1 = 625 - 25 =$ 600

$45 \cdot 43 = 45 \cdot (45-2) = 45 \cdot 45 - 45 \cdot 2 = 2025 - 90 =$ 1,935

$39 \cdot 35 = 35 \cdot (35+4) = 35 \cdot 35 + 35 \cdot 4 = 1225 + 140 =$ 1,365

$75 \cdot 77 = 75 \cdot (75+2) = 75 \cdot 75 + 75 \cdot 2 = 5625 + 150 =$ 5,775

Now take most of the steps without writing them down — if you can.

34 x 35	33 x 37	53 x 47	47 x 45	65 x 66	87 x 93	71 x 59
1225 − 35	1225 −4	2500 −9	2025 +90	4225 +65	8100 −9	4225 −36
1190	1221	2491	2115	4290	8091	4189

$35 \cdot 35 =$ 1225	$35 \cdot 36 =$ 1260	$34 \cdot 36 =$ 1224	$31 \cdot 39 =$ 1209
$45 \cdot 45 =$ 2025	$46 \cdot 45 =$ 2070	$48 \cdot 42 =$ 2016	$46 \cdot 44 =$ 2024

121

Note to accompany *Work Text* page 122

This guessing game is very much simpler than any of the games on pages 118-119. In the five tables at the end of this page, Bill's rules are: (1) multiply by 3; (2) add 10; (3) subtract from 10; (4) divide by 2; and (5) multiply the number by itself.

Notes to accompany Work Text page 123

In the "What's my Rule?" game on page 123, when Anne called 1, Bill's answer was 3. We show that Anne called 1 by starting at the tag 1. We show that Bill answered 3 by rising to the level of 3. A peg is inserted as shown (in red).

When Al called 2, Bill answered 4. We record this by beginning at the tag 2, and rising to the level of 4. A peg (shown in red) is inserted.

In the tables at the bottom of the page the rules are (1) add 3, and (2) subtract from 7. To record the

"I have a new guessing game," Bill announced one day.

"I have a secret rule. Someone calls out a number. I will use my rule and answer by saying a number. Can you guess my rule?" To himself, Bill said: whatever number anyone calls out, I will answer with a number that is 1 more.

"7," Anne called out.

"8," Bill answered.

"3," Jane called.

"4," Bill answered.

"15," Al said, "and I think you are going to answer by saying 16."

"That's right, you know my rule already. Someone call out another number."

"20," someone called out.

"How many know what my answer will be?", Bill asked.

Almost everyone did. It would be 21, because 21 is 1 more than 20.

"I have a new rule," Bill announced. To himself, he thought: whatever number anyone says, I will double it.

"2," Helen began.

"4," Bill answered.

"9," Jim called out, "and I think you will answer by saying 11." He thought Bill was adding 2 to each number called out.

"No, I'm going to say 18."

"6," Anne called out, "and I think you are going to answer by saying 12."

"Right."

By this time, everyone was sure of the secret rule.

"I have a new rule," Bill stated.

"16," Ben started off.

"8," Bill answered.

"12," Anne said and, because she thought Bill was dividing each number called out by 2, she added, "I think you plan to answer by saying 6."

"No. My answer is 4."

"18."

"10," Bill answered.

"3," Fred called out, smiling.

"Say something larger," Bill asked.

"7," Fred called out and smiled again.

"Still too small," Bill complained.

"All right, 8, then — and I'm quite sure your answer will be 0."

"Right. You have it."

"28," someone called out.

"20," Bill answered. Now everyone had discovered the secret.

No. someone calls out	3	8	1	0	11	15	19
					33	45	57

No. someone calls out	7	3	11	63	0	87	13
				73	10	97	23

No. someone calls out	1	5	7	0	15	2	9
						8	1

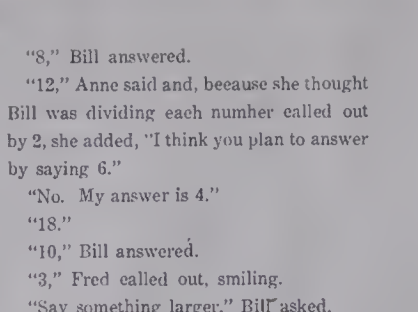
No. someone calls out	2	10	14	7	16	0	8
					8	0	4

No. someone calls out	2	4	1	3	0	5	7
					0	25	49

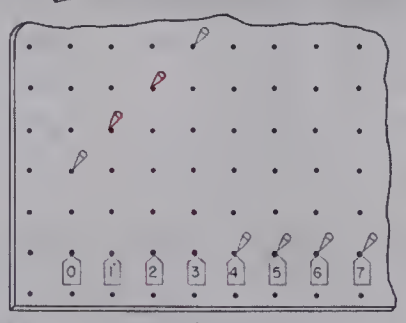
Bill found a way to keep track when playing WHAT'S MY RULE?

Using peg board, green pegs and tags, Bill worked out a way of keeping track of the numbers people call out and the number he answers after applying his rule.

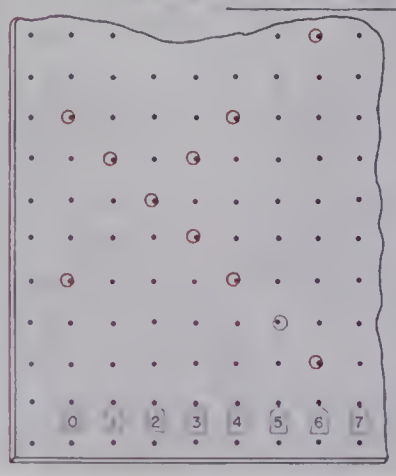
If someone calls out "2" and Bill answers "3," he finds the peg near the tag with 2 written on it. He moves the peg up to the row of holes opposite 3 on the left side of the board. It would look like the picture to the right.



"I'm thinking of a rule," Bill began.
"3," Ben called out.
"5," Bill answered. He recorded the answer. ⟶
"0," Helen called out.
"2," Bill answered, and recorded the answer. ⟶
"1," Anne called out, "and I think you are going to answer 3." ⟶
"That's right. You know my rule."
"If I say 2, will you say 4?", Al asked.
"Right. You know the rule also." ⟶



The first two results are recorded above. Can you record the others?



On the left, keep a record of the games shown below. You need to discover the secret rule in each case to finish it.

Instead of drawing pegs, circle the place where the peg would go. The first two answers are already indicated for one game, and the first answer for the second is indicated.

No. someone calls out	1	5	4	0	6	3	2
	4	8	7	3	9	6	5

No. someone calls out	5	3	6	0	4	1	2
Bill's answer	2	4	1	7	3	6	5

answers for rule (1), students can mark the appropriate peg with one color and for rule (2), they can mark the answers in another color.

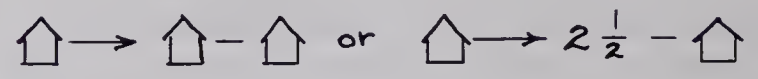
Notes to accompany *Work Text* page 124

It should be noticed that the answers based on the simple rules studied on this page form straight lines on the scoreboard and that these lines all point in the same direction.

Notes to accompany *Work Text* page 125

The answers based on the rules on this page also form straight lines on the scoreboard. The lines run either from *Southwest to Northeast* or from *Northwest to Southeast.* They move from SW to NE when the rule says add or subtract a certain number; for example:

They move from NW to SE when *the number called* is *subtracted from a fixed number,* as with

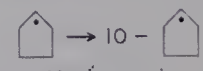


"When someone calls out a number, I imagine it as written on a tag at the bottom of the scoreboard. If my rule is to answer by saying a number that is five more than the number called out, I think this way:

"So, the green tag holds a place for any number you think of. I read that shorthand as:

'Tag goes to tag plus 5.'

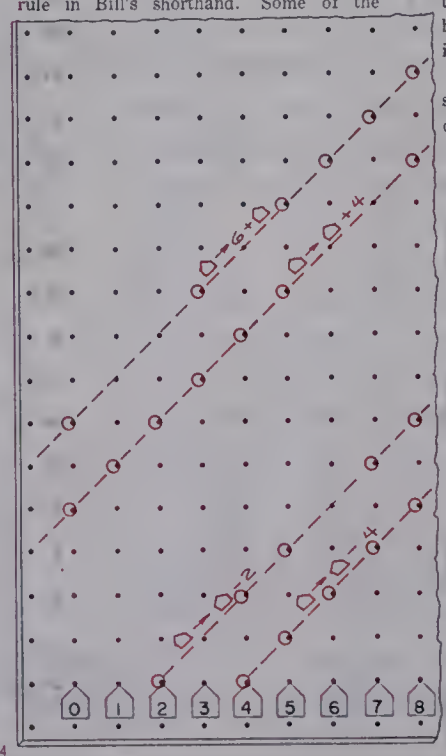
"Here's another rule. How much less than 10 is each number called out? In shorthand and words:



'Tag goes to 10 minus tag.'

Here are some games to finish. Record the results on the scoreboard and write the rule in Bill's shorthand. Some of the numbers called out and some of Bill's answers may be so large you can't record them on the small section of the scoreboard below. In that case, you will have to imagine where the pegs would go.

(Use a different color, if you can, in showing where the pegs would go for each different game.)



No. someone calls out	5	8	2	1	4	7	20
Bill's answer	3	6	0	too small	2	5	18

Bill's Rule: → − 2

Numbers called out	3	8	0	6	5	19	7
Bill's answer	9	14	6	12	11	25	13

Bill's Rule: → 6 +

Numbers called out	7	5	8	4	1	6	83
Bill's answer	3	1	4	0	too small	2	79

Bill's Rule: → − 4

Numbers called out	2	4	8	0	3	5	1
Bill's answer	6	8	12	4	7	9	5

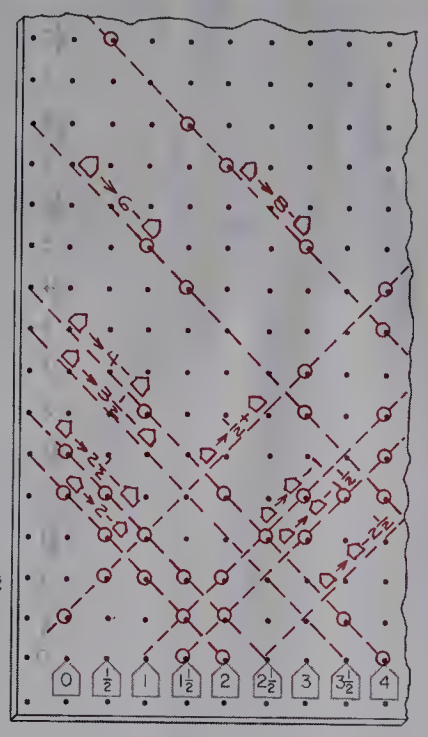
Bill's Rule: → + 4

"This far, we have been limited to whole numbers. I have changed the tags on the scoreboard so we can record some fractions . . . those fractions with denominator 2 and those that can be written with denominator 2. For example: $3\frac{1}{2}$ can be written $3 + \frac{1}{2}$ or $\frac{7}{2}$.

"Watch for patterns in the results as we record them on the scoreboard. Sometimes the patterns will be very much like those on the previous page. Some will be different.

"Here are some game starts that you can couple when you discover the secret rule."

(Remember: ⌂ is shorthand for the number someone calls out.)



⌂	Bill's answers
1	5
4	2
3	3
6	0
$1\frac{1}{2}$	$4\frac{1}{2}$

Bill's Rule: ⌂ → 6 − ⌂

⌂	Bill's answers
4	0
$2\frac{1}{2}$	$1\frac{1}{2}$
1	3
2	2
$3\frac{1}{2}$	$\frac{1}{2}$

Bill's Rule: ⌂ → 4 − ⌂

⌂	Bill's answers
$\frac{1}{2}$	1
3	$3\frac{1}{2}$
0	$\frac{1}{2}$
4	$4\frac{1}{2}$
$15\frac{1}{2}$	16

Bill's Rule: ⌂ → ½ + ⌂

⌂	Bill's answers
$\frac{1}{2}$	$1\frac{1}{2}$
$1\frac{1}{2}$	$\frac{1}{2}$
2	0
0	2
1	1

Bill's Rule: ⌂ → 2 − ⌂

⌂	Bill's answers
2	1
4	3
$1\frac{1}{2}$	$\frac{1}{2}$
3	2
$2\frac{1}{2}$	$1\frac{1}{2}$

Bill's Rule: ⌂ → ⌂ − 1

⌂	Bill's answers
0	$2\frac{1}{2}$
$\frac{1}{2}$	2
1	$1\frac{1}{2}$
$1\frac{1}{2}$	1
2	$\frac{1}{2}$

Bill's Rule: ⌂ → $2\frac{1}{2}$ − ⌂

⌂	4	½	3	0	1½	2	7½
Bill's answers	4	7½	5	8	6½	6	½

Bill's Rule: ⌂ → 8 − ⌂

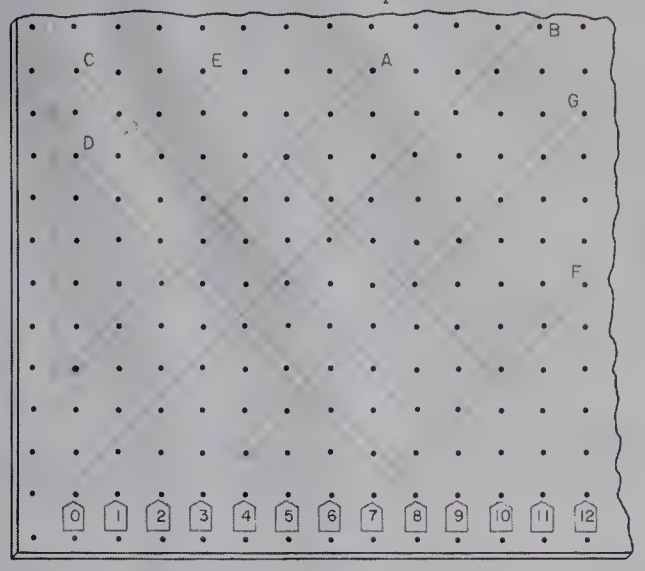
⌂	4	1½	3	2	3½	7	19
Bill's answers	2½	0	1½	½	2	5½	17½

Bill's Rule: ⌂ → ⌂ − 1½

Can you guess what pattern the following two rules will lead to?

⌂ → ⌂ − 2½ ⌂ → 3½ − ⌂

"On this scoreboard," Bill explained, "I have arranged rows of pegs to indicate results of imaginary games played. I've put a rubber band around the results of each game. Each group is lettered. I've given the rule for Game A and listed a few results. Find all the rules used and list 7 sample results."



Rule A: ⌂ → ⌂ + 3

⌂	⌂ + 3
7	10
5	8
0	3

Rule B: ⌂ → ⌂

⌂	⌂
0	0
1	1
2	2
5	5
10	10
11	11

Rule C: ⌂ → 10 − ⌂

8	2
0	10
4	6
6	4
7	3
5	5
3	7

Rule D: ⌂ → 8 − ⌂

0	8
5	3
7	1
6	2
4	4
3	5
2	6

Rule E: ⌂ → 13 − ⌂

4	9
10	3
9	4
1	12
2	11
3	10
8	5

Rule F: ⌂ → ⌂ − 7

10	3
9	2
8	1
7	0
11	4
12	5
13	6

Rule G: ⌂ → ⌂ − 3

8	5
4	1
7	4
10	7
12	9
11	8
9	6

The pegboard helps students to recognize the rule behind the numbers. (An extension of this method is used in scientific work.) The pegs also help to produce accuracy. If a student makes a mistake, he will place a peg out of line, and he will see that something is wrong.

Notes to accompany *Work Text* page 126

On pages 124 and 125, we saw how the answers to simple rules formed straight lines on the pegboard. Now the process is reversed. Rubber bands have been used to indicate the straight lines formed as a result of games played, and students are to put the pegs in. We write in the tables the numbers that describe the positions of these pegs. *After* entering these numbers, we may be able to guess the rule. This rule is then written (in shorthand) at the top of the table.

Rule B is the simplest of all. For some reason, children may find it difficult. In this rule, the number answered is *the same as* the number called. The number goes to *itself. Tag goes to tag,* is the rule.

The difficulty is probably due to the fact that, in arithmetic, we often find out the effect of adding 3 or subtracting 2. We rarely discuss leaving a number unaltered! But this rule can arise. If you travel at 1 mile an hour, how many miles do you travel in 5 hours? Answer, 5. In 8 hours? Answer, 8. The number in the answer is the same as in the question.

Notes to accompany *Work Text* page 127

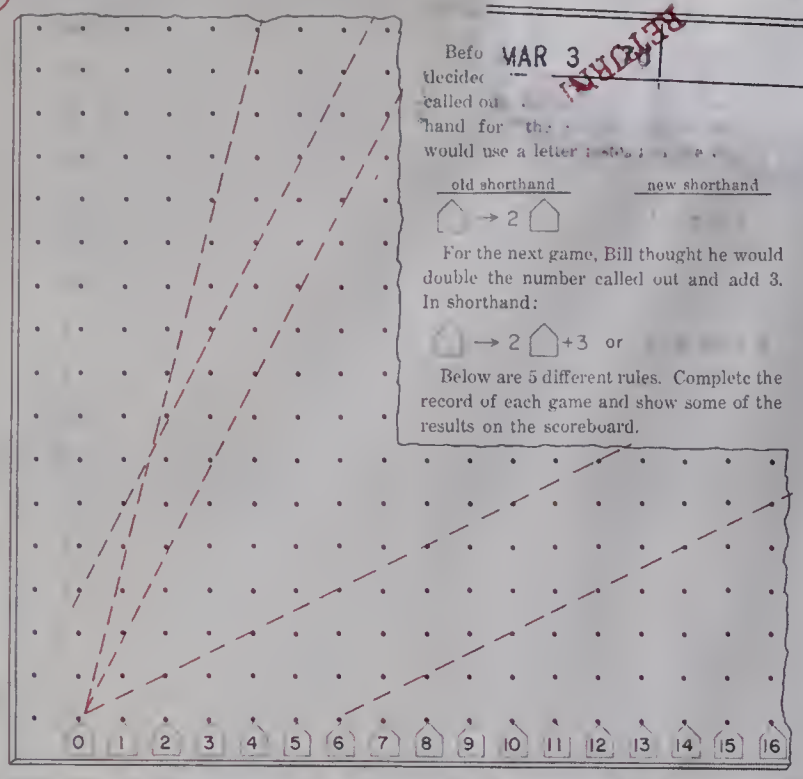
On page 127, a *tag* is replaced by a letter, such as *t* or *x*. This reminds students of the way such symbols have already been used in algebra. It does not matter which letter is used. So we use different letters in different exercises; t in one, x in another, r in a third, q in a fourth. There is no real need to use all these different letters. We use them so that students will not get any fixed idea that "the number called" must be represented by x.

In rule N, q goes to $(q \div 2) - 3$. This means *divide the number called by 2; then subtract 3.* The order is important. Parentheses are used to emphasize this. They could be omitted, because of the rule that division should be done before subtraction.

In $2x + 3$, of course, order is also important. *Double, then add 3* is the meaning. But we have already met so many expressions like $2x + 3$ that the meaning should be clear to the students.

Notice that rules J and K both have *double* in

Date Due
MAR 3



Befo
decide
called ou
hand for th
would use a letter

old shorthand → 2 new shorthand

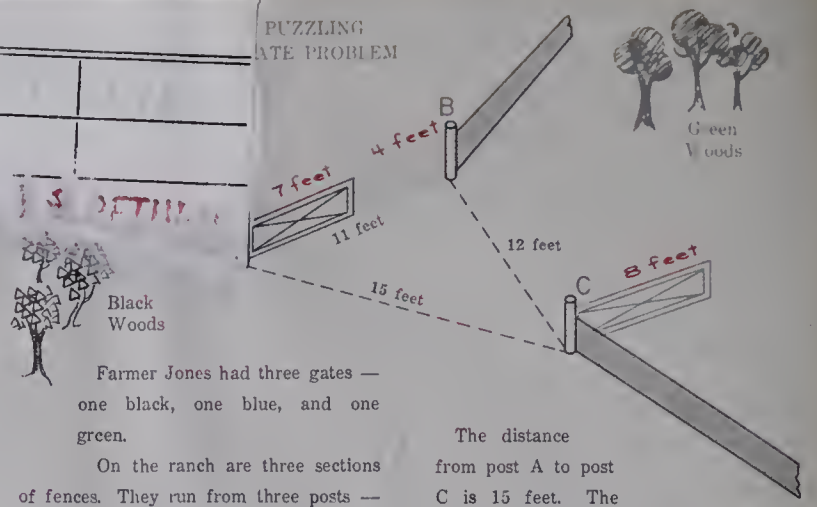
For the next game, Bill thought he would double the number called out and add 3. In shorthand:

→ 2 +3 or

Below are 5 different rules. Complete the record of each game and show some of the results on the scoreboard.

Bill's Rule J		Bill's Rule K		Bill's Rule L		Bill's Rule M		Bill's Rule N	
4	8	1	5	2	8	6	3	8	1
3	6	0	3	0	0	0	0	6	0
2	4	2	7	1	4	2	1	10	2
1	2	3	9	3	12	4	2	12	3
0	0	4	11	4	16	8	4	14	4
5	10	5	13	5	20	10	5	16	5
6	12	6	15	6	24	12	6	18	6
7	14	7	17	7	28	14	7	20	7

PUZZLING
ATE PROBLEM



Farmer Jones had three gates — one black, one blue, and one green.

On the ranch are three sections of fences. They run from three posts — A, B, and C — out to the surrounding woods.

There is a bear that roams through the woods — the black woods, the blue woods, and the green woods. When the bear comes out of the blue woods, Farmer Jones closes the black and blue gates between posts A and B (as shown above).

If the bear comes out of the green woods, he closes the blue and the green fence between posts B and C.

If the bear comes out of the black woods, Farmer Jones closes the black and the green gates — between posts A and C.

In each case, Farmer Jones moves his cattle so the bear cannot get at them — there is always a fence or gates in between.

The distance from post A to post C is 15 feet. The distance from post A to post B is 11 feet. The distance from post B to post C is 12 feet.

Each time the gates are closed, they meet without any gap between them, and they do not overlap.

How long are the gates?

The black gate is 7/4 feet long.

The green gate is 8 feet long.

Take your time. You may find this very difficult. Think about it. Make a try. Then don't think about it for a while.

Come back to it another day. Try it again.

NOTE: If you discover the answer, you must not tell others. Each must discover it by himself!

them, and the lines point in the same direction. In rules M and N, *divide by* 2 occurs.

Here is a hint for an investigation that some interested student may like to follow. By taking different rules, and drawing the corresponding lines, a student can observe how the direction and position of the line depends on the rule. With experience and practice, it is possible to look at a rule and immediately draw its line, or to look at a line and immediately give its rule.

Notes to accompany *Work Text* page 128

This puzzle may amuse the students. It can be solved by simple algebra. ("Let gate A be x feet.") *But the students do not yet have the knowledge necessary to do this.* They will probably solve it by trial and error.

Once the size of gate A has been fixed, the sizes of gates B and C follow. For example, if gate A were 10 feet, gate B would have to be 1 foot (to fill the 11 feet between posts A and B), and gate C would have to be 5 feet (to fill the 15 feet between posts A and C). But the 1-foot gate on post B and the 5-foot gate on post C would leave a big gap in the middle. So 10 feet is not the correct size for gate A. The student can next try some other size for this gate, and see whether it makes things better or worse. After a few trials, he should discover the correct solution.

2 3 4 5 6 7 8 9 10 WP 67 66 65 64 63

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